MTH4107 Introduction to Probability – 2010/11

Q1(a) By Theorem 9.1 from notes

$$\mathbb{P}(H_1 \cap F) = \mathbb{P}(F)\mathbb{P}(H_1|F) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$
$$\mathbb{P}(H_2 \cap F) = \mathbb{P}(F)\mathbb{P}(H_2|F) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$
$$\mathbb{P}(H_1 \cap H_2 \cap F) = \mathbb{P}(F)\mathbb{P}(H_1 \cap H_2|F) = \frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$$

(b) We have

$$\mathbb{P}(H_1 \cap F)\mathbb{P}(H_2 \cap F) = \frac{1}{16} \neq \frac{1}{8} = \mathbb{P}(H_1 \cap H_2 \cap F).$$

Hence $H_1 \cap F$ and $H_2 \cap F$ are not independent.

Q2. Let E_0, E_1, E_2 be the events "neither player recovers", "exactly one player recovers" and "both players recover" respectively. Let W be the event "the match is won". By the Theorem of Total Probability (Theorem 9.3)

$$\mathbb{P}(W) = \mathbb{P}(W|E_0)\mathbb{P}(E_0) + \mathbb{P}(W|E_1)\mathbb{P}(E_1) + \mathbb{P}(W|E_2)\mathbb{P}(E_2).$$

Since the recoveries of the two players are independent we have

$$\mathbb{P}(E_0) = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$$
$$\mathbb{P}(E_1) = 2\left(\frac{1}{3}\right)\left(\frac{2}{3}\right) = \frac{4}{9}$$
$$\mathbb{P}(E_2) = \left(\frac{1}{3}\right)^2 = \frac{1}{9}.$$

We are told the relevant conditional probabilities in the question and so

$$\mathbb{P}(W) = \frac{1}{16} \times \frac{4}{9} + \frac{1}{2} \times \frac{4}{9} + \frac{3}{4} \times \frac{1}{9} = \frac{1}{3}.$$

Q3. Consider picking a voter at random with all choices equally likely. Let O be the event "the chosen voter voted for Obama" and U be the event "the chosen voter was aged under 30".

(a) By the Theorem of Total Probability (Theorem 9.2) using the partition of S into the two events U and U^c

$$\mathbb{P}(O) = \mathbb{P}(O|U)\mathbb{P}(U) + \mathbb{P}(O|U^c)\mathbb{P}(U^c)$$

The question asks for $\mathbb{P}(O|U^c)$. Putting in the values from the question

$$\frac{52.6}{100} = \frac{66}{100} \times \frac{18}{100} + \mathbb{P}(O|U^c) \times \left(1 - \frac{18}{100}\right)$$

which gives $\mathbb{P}(O|U^c) = 0.497$ to 3 decimal places.

(b) This asks for $\mathbb{P}(U|O)$. By Bayes Theorem (Theorem 9.4)

$$\mathbb{P}(U|O) = \frac{\mathbb{P}(O|U)\mathbb{P}(U)}{\mathbb{P}(O)} = \frac{\frac{66}{100} \times \frac{18}{100}}{\frac{52.6}{100}} = 0.226$$

to 3 decimal places.

Q4*. Let P_1 be the event that the first test is positive and H be the event that the person has the disease.

(a) By the Theorem of Total Probability

$$\begin{aligned} \mathbb{P}(P_1) &= \mathbb{P}(P_1|H)\mathbb{P}(H) + \mathbb{P}(P_1|H^c)\mathbb{P}(H^c) \\ &= \frac{9}{10} \times \frac{1}{10} + \frac{1}{100} \times \frac{9}{10} \\ &= \frac{99}{1000}. \end{aligned}$$

(b) By Bayes Theorem

$$\begin{aligned} \mathbb{P}(H|P_1) &= \frac{\mathbb{P}(P_1|H)\mathbb{P}(H)}{\mathbb{P}(P_1)} \\ &= \frac{9}{10} \times \frac{1}{10} \times \frac{1000}{99} \\ &= \frac{10}{11}. \end{aligned}$$

(c) Let P_2 be the event that the second test is positive. By the Theorem of Total Probability for conditional probability

$$\begin{aligned} \mathbb{P}(P_2|P_1) &= \mathbb{P}(P_2|H \cap P_1)\mathbb{P}(H|P_1) + \mathbb{P}(P_2|H^c \cap P_1)\mathbb{P}(H^c|P_1) \\ &= \frac{9}{10} \times \frac{10}{11} + \frac{1}{100} \times \frac{1}{11} \\ &= \frac{901}{1100}. \end{aligned}$$

AQ1 (i) 1/2 (ii) 1/6

AQ2 (ii) 0.12

Please let me know if you have any comments or corrections