MTH4107 Introduction to Probability – 2010/11

Exercise Sheet 5

These questions are designed to help you understand the material covered in week 5 lectures. You should write up your solution to the starred question, Q4, **clearly** *and hand it in during your week 8 exercise class for feedback. Put your* **full name and student number** *on the top of your solution. It is important that you make a serious attempt to do* **all** *of questions Q1-Q4 before week 8 lectures begin. Questions AQ1-AQ3 are for additional practice. You should attempt them when you have time.*

In addition to your lecture notes material relating to these questions can be found in Devore, Chapter 2 Sections 2.4 and 2.5, or Ross, Chapter 3 Sections 3.2 and 3.4.

Q1. A couple wants two girls. They decide to keep having children and stop either when they have two girls or when they have four children, whichever happens first. Assume that they do not have twins, or other multiple births and that each child they have is equally likely to be a boy or a girl independently of all other children.

(a) Write down the sample space.

(b) Are all elements of the sample space equally likely? Justify your answer.

(c) Calculate the probability that they have two girls.

(d) Calculate the conditional probability that they have two girls given that their first child is a boy.

Q2. An element from the set $\{1, 2, 3, \dots, 36\}$ is chosen at random with all choices equally likely. Let E be the event "the chosen integer is even", O be the event "the chosen integer is odd", and Q be the event "the chosen integer is a perfect square".[1]

(a) Are the events E and O independent? Justify your answer.

(b) Are the events E and Q independent? Justify your answer.

(c) Are the events O and Q independent? Justify your answer.

Q3. The top card of a thoroughly shuffled deck of playing cards[2] is turned over. Let A be the event "the card is an Ace", R be the event "the card belongs to a red suit ($\diamondsuit$ or $\heartsuit$)", and M be the event "the card belongs to a major suit ($\heartsuit$ or $\spadesuit$)". Show that the events A, R and M are mutually independent.

[1] An integer is a *perfect square* if it is equal to the square of an integer.

[2] A deck of playing cards is made up of 52 cards split into 4 suits ($\clubsuit, \diamondsuit, \heartsuit, \spadesuit$) with each suit made up of one card of each of 13 ranks ($2, 3, 4, \dots, 10$,Jack,Queen,King,Ace).

Q4* (a) [Q1 continued]

(i) Find the conditional probability that the couple's first child is a boy given that they have two girls.

(ii) Find the conditional probability that their first child is a girl given that they have two girls.

(b) Let A and B be events in a sample space S. Prove that

$$\mathbb{P}(A^c|B) = 1 - \mathbb{P}(A|B).$$

Deduce that, if A and B are independent, then A^c and B are also independent. Which question on this exercise sheet illustrates these statements?

AQ1. You are allowed up to three attempts to pass the Probability exam. Suppose that your probability of passing is p at each attempt, independent of all other attempts. How big must p be so that the probability that you eventually pass the exam is at least 0.9? Give your answer to 4 decimal places.

AQ2. There are two roads from A to B and two roads from B to C. Suppose that each road is closed with probability p and that the state of each road is independent of the others. What condition on p ensures that the probability that I cannot travel from A to C is at most $1/2$?

AQ3. Let $n \geq 2$ be an integer. Use induction to prove that n can be expressed as a product of prime numbers.