MTH4107 Introduction to Probability – 2010/11

Solutions to Exercise Sheet 3

Q1 The key here is to remember the how to interpret the set operations $\cup$, $\cap$, and $\setminus$ in words.

(a) (i) $G \cap H$

(ii) $(F \cap G) \setminus H$ (or $F \cap G \cap H^c$)

(iii) $(F \cap G \cap H^c) \cup (F \cap G^c \cap H) \cup (F^c \cap G \cap H)$ (there are other possibilities),

(b) (i) This is the event "the chosen student does not speak Hungarian".

(ii) This is the event "the chosen student speaks none of the languages"

(iii) This is also the event "the chosen student speaks none of the languages" (notice how the same event can be expressed in ways that look different).

(c) The first sentence means that $|S| = 100, |G| = 30$. The second sentence means that $F \cap G = \emptyset$ and $H \subset G$. Again there are equivalent ways of expressing this, for example $F \subset G^c$ and $H \cap G^c = \emptyset$

Q2

(a) $\mathbb{P}(A^c) = 1 - \mathbb{P}(A) = 3/5$ (by Proposition 5.1)

(b) For this question we need the inclusion-exclusion formula for two events (Proposition 5.6).

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) = 2/5 + 1/2 - 3/20 = 3/4$$

(c) Since B is the disjoint union of $A \cap B$ and $A^c \cap B$, Axiom 3 gives $\mathbb{P}(B) = \mathbb{P}(A \cap B) + \mathbb{P}(A^c \cap B)$. Thus

$$\mathbb{P}(A^c \cap B) = \mathbb{P}(B) - \mathbb{P}(A \cap B) = 1/2 - 3/20 = 7/20$$

(d) $B \setminus A = A^c \cap B$ and so this probability is the same as that calculated in part (c). Hence $\mathbb{P}(B \setminus A) = 7/20$.

Q3* (a) Let S be the set of all books, A be the set of non-fiction books and B be the set of hardback books. Since the book is selected at random from S we have $\mathbb{P}(A) = |A|/|S| = 3/10$, $\mathbb{P}(B) = |B|/|S| = 7/10$, and $\mathbb{P}(A \cap B) = |A \cap B|/|S| = 1/5$.

(i) This asks for $\mathbb{P}(A^c)$. We have $\mathbb{P}(A^c) = 1 - \mathbb{P}(A) = 7/10$

(ii) Here we want $\mathbb{P}(A \cup B)$. We have

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) = 3/10 + 7/10 - 1/5 = 4/5$$

(iii) We want $\mathbb{P}(A^c \cap B)$. By a similar argument to Q2(c) we have
$\mathbb{P}(A^c \cap B) = \mathbb{P}(B) - \mathbb{P}(A \cap B) = 7/10 - 1/5 = 1/2$.

(b)

(i) **Proof** By Proposition 5.6, $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$. Hence

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B).$$

Since $\mathbb{P}(A \cup B) \leq 1$ by Corollary 5.4, we have

$$\mathbb{P}(A \cap B) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1.$$

(ii) **Proof** Since $A \triangle B$ is the disjoint union of $A \setminus B$ and $B \setminus A$, Axiom 3 gives

$$\mathbb{P}(A \triangle B) = \mathbb{P}(A \setminus B) + \mathbb{P}(B \setminus A). \tag{1}$$

Similarly

$$\mathbb{P}(A) = \mathbb{P}(A \setminus B) + \mathbb{P}(A \cap B) \tag{2}$$

and

$$\mathbb{P}(B) = \mathbb{P}(B \setminus A) + \mathbb{P}(A \cap B). \tag{3}$$

Substituting (2) and (3) into (1) we obtain

$$\mathbb{P}(A \triangle B) = \mathbb{P}(A) + \mathbb{P}(B) - 2\mathbb{P}(A \cap B).$$

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OR

Proof Since $A \cup B$ is the disjoint union of $A \triangle B$ and $A \cap B$, Axiom 3 gives

$$\mathbb{P}(A \cup B) = \mathbb{P}(A \triangle B) + \mathbb{P}(A \cap B). \tag{4}$$

By Proposition 5.6

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B). \tag{5}$$

Substituting (5) into (4) and rearranging terms we obtain

$$\mathbb{P}(A \triangle B) = \mathbb{P}(A) + \mathbb{P}(B) - 2\mathbb{P}(A \cap B). \tag{6}$$

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Please let me know if you have any comments or corrections