MTH4107 Introduction to Probability – 2010/11

Exercise Sheet 11

These questions are designed to help you understand cumulative distribution functions and continuous random variables. It is important that you make a serious attempt to do questions Q1-Q3 before the second semester starts. Questions AQ1-2 are for additional practice. You should attempt it when you have time.
In addition to your lecture notes, material relating to these questions can be found in Devore, Chapter 4 Sections 4.1-2. or Ross, Chapter 5 Sections 5.1-3.

Q1. Let S be the right-angled triangle with vertices $(0,0),(1,0),(1,1)$. Let p be a point chosen at random from within the triangle with the probability that p is in any fixed region being proportional to the area of the region. Let T be the random variable "the angle that the line from $(0,0)$ to p makes with the x-axis". Find the cumulative distribution function and probability density function of T.

Q2(a) Let U be a random variable with the Uniform$[a,b]$ distribution. Show that $\mathrm{E}(U) = (b+a)/2$ and $\mathrm{Var}(U) = (b-a)^2/12$.
(b) Let X be a random variable with the Exponential(λ) distribution. Show that $\mathrm{Var}(X) = 1/\lambda^2$.

Q3. Let U be a random variable with the Uniform$[0,1]$ distribution. Show that $U+1$ also has a uniform distribution but that U^2 does not. (Hint: calculate the cumulative distribution functions of $U+1$ and U^2.)

AQ1.(a) Let X be a random variable with the Exponential(1) distribution. Show that for $y, z > 0$, the conditional probability that $X > y + z$ given that $X > y$ is equal to the probability that $X > z$. This is called the memoryless property of the exponential distribution; can you see why?
(b) Let U be a random variable with the Uniform$[0,1]$ distribution. Show that for $0 < y < 1$, $0 < z < 1$, the conditional probability that $U > y + z$ given that $U > y$ is strictly less than the probability that $U > z$.

AQ2. Let S be the unit square with vertices $(0,0),(1,0),(0,1),(1,1)$. Let p be a point chosen randomly from within the square with the probability that p is in any fixed region being proportional to the area of the region. Let T be the random variable "the sum of the $x-$coordinate of p and the $y-$coordinate of p". Find the cumulative distribution function and probability density function of T.