MTH4107 Introduction to Probability

End-term Test Solutions

1.

(a) (BOOKWORK) Suppose A and B are events with $\mathbb{P}(A) > 0$ and $\mathbb{P}(B) > 0$. Then

$$\mathbb{P}(B|A) = \mathbb{P}(A|B) \times \frac{\mathbb{P}(B)}{\mathbb{P}(A)}.$$

(b) Let H be the event the person has the disease and P be the event the test is positive.

(i) (SIMILAR TO CW)

$$\mathbb{P}(P) = \mathbb{P}(P|H)\mathbb{P}(H) + \mathbb{P}(P|H^c)\mathbb{P}(H^c) = \frac{9}{10} \times \frac{1}{100} + \frac{1}{9} \times \frac{99}{100} = \frac{119}{1000}$$

(ii) (SIMILAR TO CW)

$$\mathbb{P}(H|P) = \mathbb{P}(P|H) \times \frac{\mathbb{P}(H)}{\mathbb{P}(P)} = \frac{9}{10} \times \frac{1}{100} \times \frac{1000}{119} = \frac{9}{119}$$

2.

(a) (BOOKWORK) The *expected value* of X is given by

$$\mathbb{E}(X) = \sum_{r \in Range(X)} r\mathbb{P}(X = r).$$

Suppose $\mathbb{E}(X) = \mu$. Then the *variance* of X is given by

$$\mathrm{Var}(X) = \sum_{r \in Range(X)} (r - \mu)^2 \mathbb{P}(X = r).$$

(b) (i)

m	0	1	2
$\mathbb{P}(X = m)$	1/9	3/9	5/9

(ii) $\mathbb{E}(X) = 0 \times (1/9) + 1 \times (3/9) + 2 \times (5/9) = 13/9$
$\mathbb{E}(X^2) = 0 \times (1/9) + 1 \times (3/9) + 4 \times (5/9) = 23/9$ so
$\mathrm{Var}(X) = (23/9) - (13/9)^2 = 38/81$.

3.

(a) (BOOKWORK) X has the Poisson(λ) distribution.
Suppose that, on average, I receive λ telephone calls in a unit of time. Let X be the actual number of calls I receive in a given time interval of unit length. Then X has Poisson(λ) distribution.

(b) (BOOKWORK)

$$\begin{aligned}
\mathbb{E}(X) &= \sum_{m=0}^{\infty} mP(X = m) \\
&= \sum_{m=1}^{\infty} me^{-\lambda}\lambda^m/m! \\
&= \lambda e^{-\lambda}\sum_{k=0}^{\infty} \lambda^k/k! \\
&= \lambda.
\end{aligned}$$

since $\sum_{k=0}^{\infty} \lambda^k/k! = e^{\lambda}$

4. (SIMILAR TO CW)

(a)

m	0	1	2
$\mathbb{P}(X = m)$	5/18	4/9	5/18
$\mathbb{P}(Y = m)$	5/18	4/9	5/18

(b) $\mathbb{E}(X) = 4/9 + 5/9 = 1 = \mathbb{E}(Y)$.
$\mathbb{E}(XY) = 1/9 + 1/3 + 1/3 + 4/9 = 11/9$ so $\text{Cov}(X, Y) = 11/9 - 1 = 2/9$.

(c) X and Y are not independent since $\text{Cov}(X, Y) \neq 0$ (or give a direct proof).