1 Introduction and Basic Concepts

Graph Theory is a new and active branch of mathematics. Its origins can be traced back to Euler's work on the Königsberg bridges problem in 1766. However, a comprehensive theory only began to develop in the last hundred years. The first book on graph theory, Theorie der endlichen und unendlichen Graphen by D. König, was published in 1935. Since the 1940s, a major stimulus for the development of graph theory has been to provide efficient algorithms for solving network problems in operational research.

This module is intended to be an introduction to the theory of graphs and how it can be used to solve optimization problems in networks. One of the most appealing aspects of graph theory is that abstract ideas in the theory can be illustrated by drawings. You should develop the habit of drawing pictures of graphs to investigate and illustrate the problems, concepts and algorithms given in the course.

1.1 Formal definition of a graph

A *graph* G is an ordered pair $(V(G), E(G))$ where $V(G)$ and $E(G)$ are disjoint sets, called the *vertices* and *edges* of G, respectively, together with an incidence function f which associates an unordered pair of (not necessarily distinct) vertices $\{u, v\}$ with each edge e of G. When $f(e) = \{u, v\}$ we shall say that:

- e is *incident* with u and v,
- u and v are the *end vertices* of e, and
- u and v are *adjacent*.

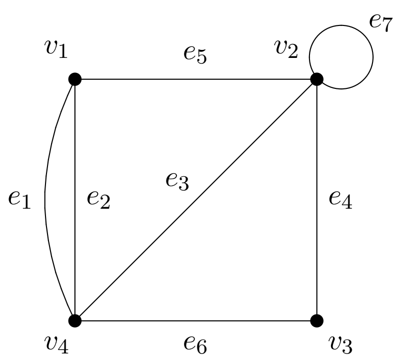
Henceforth we shall use the above three statements rather than referring explicitly to the incidence function.

1.2 Pictorial Representation of a Graph

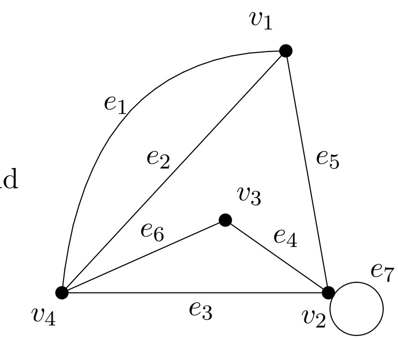
We represent the graph G by a drawing in the plane with the vertices represented by points and each edge e by a line between the two points corresponding to its end vertices.

Example 1.2 Let G be the graph with the following formal definition: $V(G) = \{v_1, v_2, v_3, v_4\}$, $E(G) = \{e_1, e_2, e_3.e_4, e_5, e_6\}$, and incidence function f given by $f(e_1) = \{v_1, v_4\}, f(e_2) = \{v_1, v_4\}, f(e_3) = \{v_2, v_4\}, f(e_4) =$

$\{v_2, v_3\}, f(e_5) = \{v_1, v_2\}, f(e_6) = \{v_3, v_4\}, f(e_7) = \{v_2, v_2\}$. Then two possible pictorial representations for G are:



and



1.3 Matrix Representation of a Graph

We represent the graph G by a matrix $M(G)$ with rows indexed by $V(G)$ and columns indexed by $E(G)$. For v a vertex of G and e an edge of G, the entry in the row labelled by v and column labelled by e is equal to 2 if both end vertices of e are equal to v, is equal to 1 if exactly one end vertex of e is equal to v and is equal to 0 otherwise. The matrix $M(G)$ is called the *incidence matrix* of G.

Example 1.3 Let G be the graph from Example 1.2 Then the incidence matrix of G is:

$$M(G) = \begin{array}{c|cccccc} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ v_1 & 1 & 1 & 0 & 0 & 1 & 0 \\ v_2 & 0 & 0 & 1 & 1 & 1 & 0 \\ v_3 & 0 & 0 & 0 & 1 & 0 & 1 \\ v_4 & 1 & 1 & 1 & 0 & 0 & 1 \end{array}$$

1.4 Definition

A *loop* in a graph G is an edge with both its end vertices the same. A *multiple edge* in G is a set of two or more edges with the same end vertices. We say that G is *simple* if G has no loops and no multiple edges.

1.5 Definitions

A *walk* in a graph is an alternating sequence of vertices and edges such that each edge is preceded by one of its end vertices and followed by its other end vertex. Given a walk $W = x_0e_1x_1e_2 \ldots e_mx_m$ we say that W is a *walk from x_1 to x_m*. Given a walk W_1 from u to v and a walk W_2 from v to x, we use W_1W_2 to denote the walk from u to x we obtain by first traversing W_1 then W_2. A *trail* is a walk which does not repeat edges. A *path* is a walk which does not repeat vertices or edges. A walk is *closed* if it starts and ends with the same vertex. A *tour* is a closed trail. A *cycle* is a tour which contains at least one edge and does not repeat vertices, except when the first vertex is repeated as the last vertex. The *length* of a walk is equal to the number of edges in the sequence, counting repeated edges the appropriate number of times. Thus, in Example 1.2 we have $W = v_1e_5v_2e_5v_1e_2v_4$ is a walk of length three, $T = v_1e_5v_2e_7v_2e_3v_4e_2v_1$ is a tour of length four, $P = v_1e_5v_2e_3v_4$ is a path of length two, and $C = v_1e_5v_2e_3v_4e_2v_1$ is a cycle of length three. If G is simple then we can unambiguously represent a walk as a sequence of vertices. In particular, we can uniquely represent an edge e of a simple graph by its pair of end vertices and write $e = uv$.

1.6 Definition

A *directed graph* or *digraph* D is a graph in which each edge e has been given a fixed direction from one end vertex u to its other end vertex v. We call directed edges *arcs*, and use $A(D)$ to denote the set of arcs of D. We say that

- e is an arc *from u to v*, and
- u is the *tail* of e and v is the *head* of e.

A *directed walk* in a digraph D is a walk in which each arc is preceded by its tail and followed by its head.

1.7 Definition

A *network* is a graph or digraph in which to each edge e we associate a real number $w(e)$ called the *weight* of e. The *length* of a walk in a network is the sum of the weights of its edges, counting repeated edges the appropriate number of times.

1.8 Definitions

A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$ and each $e \in E(H)$ has the same end vertices in H and G. If H is a subgraph of G then we say H is a *proper subgraph* of G if $H \neq G$ and that H is a *spanning subgraph* of G if $V(H) = V(G)$. Let G_1 and G_2 be two subgraphs of G. Then

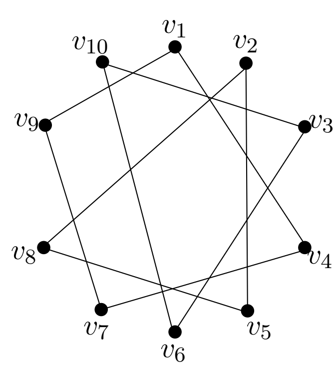
- $G_1 \cup G_2$ denotes the subgraph of G with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$.
- $G_1 \cap G_2$ denotes the subgraph of G with $V(G_1 \cap G_2) = V(G_1) \cap V(G_2)$ and $E(G_1 \cap G_2) = E(G_1) \cap E(G_2)$.

1.9 Definition

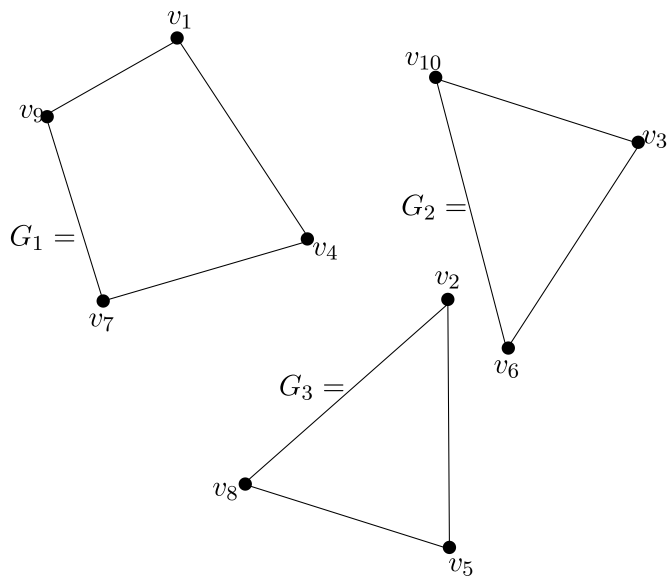
A graph G is *connected* if every pair of vertices of G are joined by a walk. The *connected components* of a graph G are the maximal connected subgraphs of G, where a connected subgraph is *maximal* if it is not a proper subgraph of any other connected subgraph of G.

Example 1.9 Consider the following graph G.

$G =$



Then G has the three connected components G_1, G_2 and G_3 given below.



We will show that every graph is the disjoint union of its connected components. We need the following lemma.

1.10 Lemma

Let G_1, G_2 be connected subgraphs of a graph G such that $V(G_1) \cap V(G_2) \neq \emptyset$. Then $G_1 \cup G_2$ is connected.

Proof Suppose $x \in V(G_1) \cap V(G_2)$. Choose vertices u and v of $G_1 \cup G_2$. If u and v both belong to G_1 (or G_2) then, since G_1 is connected, we can find a walk in G_1 joining u and v. On the other hand, if $u \in V(G_1)$ and $v \in V(G_2)$ then we can find a walk W_1 in G_1 from u to x, and a walk W_2 in G_2 from x to v. Then W_1W_2 is a walk from u to v in $G_1 \cup G_2$. Thus, in all cases, $G_1 \cup G_2$ contains a walk from u to v. Since u and v are arbitrary vertices of $G_1 \cup G_2$, it follows that $G_1 \cup G_2$ is connected.

1.11 Lemma

Let $G_1, G_2, \dots, G_m$ be the connected components of a graph G. Then
(a) $\{V(G_1), V(G_2), \dots V(G_m)\}$ is a partition of $V(G)$ and

(b) $\{E(G_1), E(G_2), \ldots, E(G_m)\}$ is a partition of $E(G)$.

Proof (a) Since each vertex of G belongs to a connected subgraph, each vertex belongs to a connected component and hence we have

$$\bigcup_{i=1}^{m} V(G_i) = V(G).$$

Suppose $V(G_i) \cap V(G_j) \neq \emptyset$ for some $1 \leq i < j \leq m$. Then $G_i \cup G_j$ is connected by Lemma 1.10. This contradicts the fact that G_i and G_j are supposed to be connected components, and hence *maximal* connected subgraphs, of G. The only alternative is that $V(G_i) \cap V(G_j) = \emptyset$. Thus each pair of connected components of G are vertex disjoint and $\{V(G_1), V(G_2), \ldots, V(G_m)\}$ is a partition of $V(G)$.

(b) Since each edge belongs to a connected subgraph of G we have

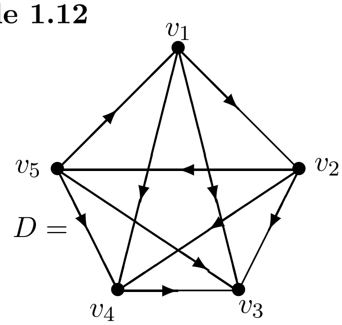
$$\bigcup_{i=1}^{m} E(G_i) = E(G).$$

Suppose $e \in E(G_i) \cap E(G_j)$ for some $1 \leq i < j \leq m$. Let u be an end vertex of e. Then $u \in V(G_i) \cap V(G_j)$. This contradicts (a). Thus any two connected components of G are edge-disjoint and $\{E(G_1), E(G_2), \ldots, E(G_m)\}$ is a partition of $E(G)$.

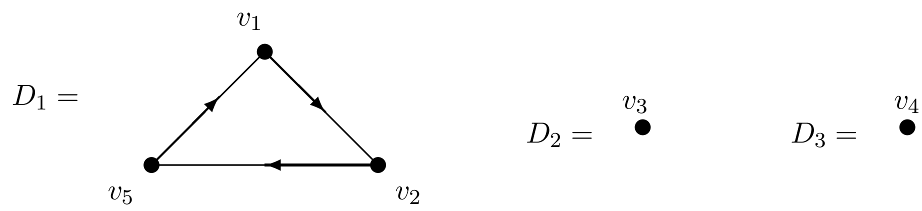
1.12 Definition

A digraph D is *strongly connected* if, for each ordered pair of vertices u and v of D, there is a directed walk from u to v in D. The *strongly connected components* of D are the maximal strongly connected subdigraphs of D.

Example 1.12



D is not strongly connected since there is no directed walk from v_3 to v_2. It has the following three strongly connected components:



We can show that the strongly connected components of a digraph are pairwise disjoint in a similar way as we did for graphs.

1.13 Lemma

Let D_1, D_2 be stongly connected subdigraphs of a digraph D such that $V(D_1) \cap V(D_2) \neq \emptyset$. Then $D_1 \cup D_2$ is strongly connected.

Proof Exercise.

1.14 Lemma

Let $D_1, D_2, \ldots, D_m$ be the strongly connected components of a digraph D. Then $\{V(D_1), V(D_2), \ldots V(D_m)\}$ is a partition of $V(D)$.

Proof Exercise.

1.15 Remark

It is not true in general that $\{A(D_1), A(D_2), \ldots, A(D_m)\}$ is a partition of $A(D)$ since D may have arcs which belong to none of its strongly connected components, see Example 1.12.

1.16 Lemma

(a) Let G be a graph and u, v be distinct vertices of G. If G has a walk from u to v then G has a path from u to v.

(b) Let D be a digraph and u, v be distinct vertices of D. If D has a directed walk from u to v then D has a directed path from u to v.

Proof (a) Choose a walk W from u to v in G such that W is as short as possible. Let $W = v_0 e_1 v_1 \dots e_m v_m$, where $v_0 = u$ and $v_m = v$. Suppose W is not a path. Then either some edge or some vertex is repeated in W. If $e_i = e_j$ for some $1 \leq i < j \leq m$ then we have $\{v_{i-1}, v_i\} = \{v_{j-1}, v_j\}$. So either $v_i = v_j$, or $v_i = v_{j-1}$ and $v_{i-1} = v_j$. In both cases we also have some vertex repeated in W. Thus we may assume that $v_i = v_j$ for some $0 \leq i < j \leq m$. Let $W_1 = v_0 e_1 v_1 \dots e_i v_i$ be the section of W from v_1 to v_i and $W_2 = v_j e_{j+1} v_{j+1} \dots e_m v_m$ be the section of W from v_j to v_m. Then $W_1 W_2$ is a walk from $v_0 = u$ to $v_m = v$ in G which is shorter than W. This contradicts the choice of W. Hence W must be a path.

(b) The proof is similar to that of (a),

1.17 Corollary

(a) A graph G is connected if and only if all pairs of vertices of G are joined by a path.

(b) A digraph D is connected if and only if all ordered pairs of vertices of D are joined by a directed path.

Proof This follows immediately from Lemma 1.16.

1.18 Definition

Let G be a graph and $v \in V(G)$. The *degree* of v is the number of edges of G incident with v, counting each loop twice. We denote this by $d_G(v)$, or $d(v)$ when it is obvious to which graph we are referring.

1.19 Lemma (The handshaking lemma)

In every graph G we have $\sum_{v \in V(G)} d(v) = 2|E(G)|$.

Proof Let e be an edge of G with end vertices x and y. If e is not a loop then $x \neq y$ and e contributes one to $d(x)$ and one to $d(y)$. Thus e contributes two to $\sum_{v \in V(G)} d(v)$. If e is a loop then $x = y$ and e contributes exactly two to both $d(x)$ and to $\sum_{v \in V(G)} d(v)$. Thus all edges of G contribute 2 to $\sum_{v \in V(G)} d(v)$, and hence $\sum_{v \in V(G)} d(v) = 2|E(G)|$.

1.20 Definition

Let D be a digraph and $u \in V(D)$. Then the *out-degree* of u is the number of arcs in D with tail u. We denote this by $d_D^+(u)$ or more simply by $d^+(u)$. We give similar definitions for the in-degree of u, $d_D^-(u)$ or $d^-(u)$.

1.21 Lemma

For any digraph D we have $\sum_{v \in V(D)} d^+(u) = |A(D)| = \sum_{v \in V(D)} d^-(u)$.

Proof Exercise.