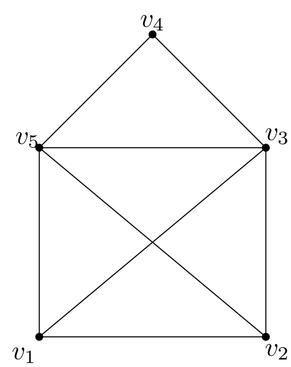
7 Euler Tours

7.1 Definition

Recall that a *trail* in a graph G is walk which does not repeat edges, and a *tour* is a closed trail. An *Euler tour*, respectively *Euler trail*, of G is a tour, respectively trail, which uses every vertex of G at least once and every edge of G exactly once. We shall say that G is *Eulerian* if G has an Euler tour.

7.2 Example

Consider the following graph G.



Then $R = v_1v_2v_3v_4v_5v_1v_3v_5v_2$ is an Euler trail for G, but G has no Euler tour.

The following result determines which graphs are Eulerian. The 'Necessity' part of the proof was given by Euler in 1736.

7.3 Theorem

Let G be a graph. Then G is Eulerian if and only if G is connected and all vertices of G have even degree.

Proof

Necessity Suppose G has an Euler tour R. Since every vertex of G belongs to R we can walk from any vertex to any other vertex along R and hence G is connected. Choose $v \in V(G)$. Since R enters v the same number of times that it leaves v and uses each edge of G which is incident to v exactly once, v must have even degree in G.

Sufficiency Suppose G is connected and $d_G(v)$ is even for all $v \in V(G)$. We show that G has an Euler tour by induction on $|E(G)|$.

Base Case. Suppose $|E(G)| = 0$. Since G is connected G only has one vertex, v say, and $R = v$ is an Euler tour of G.

Induction Hypothesis. Suppose $|E(G)| > 0$ and that all graphs which are connected, have all vertices of even degree, and have fewer edges than G, are

Eulerian.
Inductive Step Since $|E(G)| > 0$ and $d_G(v)$ is even for all $v \in V(G)$, G is not a tree. Thus G contains a cycle. Let R be a tour of maximum length in G. Then $|E(R)| > 0$.

Suppose R is not an Euler tour of G. Then $E(R) \neq E(G)$. Let $H = G - E(R)$. Then $E(H) \neq \emptyset$. Choose a component H_1 of H such that $E(H_1) \neq \emptyset$. Since G is connected, some vertex v_1 of H_1 is also a vertex of R. Furthermore, $d_{H_1}(v) = d_G(v) - d_R(v)$ for all $v \in V(H_1)$ and since $d_G(v)$ and $d_R(v)$ are even, we have $d_{H_1}(v)$ is even, Thus H_1 is connected and has all vertices of even degree. Since $|E(H_1)| < |E(G)|$ it follows by induction that H_1 has an Euler tour R_1. Let R' be the tour of G obtained by starting at v_1 and first traversing R and then traversing R_1. Clearly R' is longer than R. This contradicts the choice of R as a longest tour in G.

The only way out of this contradiction is that R is an Euler tour of G, and hence G is Eulerian.

7.4 Corollary

Let G be a graph and x and y be vertices of G. Then G has an Euler trail from x to y if and only if
(a) G is connected,
(b) $d_G(x)$ and $d_G(y)$ are odd, and
(c) $d_G(v)$ is even for all $v \in V(G) - \{x, y\}$.

Proof Let H be the graph obtained from G by adding a new edge e with end vertices x and y. Then G has an Euler trail from x to y if and only if H has an Euler tour. Applying Theorem 7.3 to H we deduce that the corollary holds.

7.5 Algorithm to construct an Euler tour

The proof of Theorem 7.3 is constructive and gives rise to the following polynomial algorithm for constructing an Euler tour. Suppose G is a connected graph such that all vertices of G have even degree.

Initial Step Construct a tour R_1 in G by constructing a maximal trail starting at some vertex x_1: we walk along edges of G starting at x_1 and choosing a new edge every time we leave a vertex until no more new edges can be chosen. Since all vertices of G have even degree this maximal trail must end at x_1 and hence it is a tour.

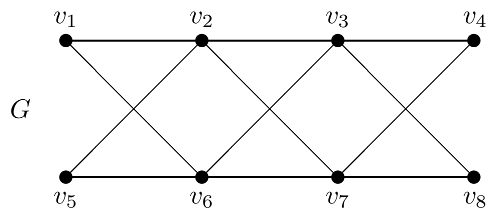
Iterative Step Suppose we have constructed a tour R_i in G for some $i \geq 1$.

- If R_i is not an Euler tour then choose a component H of $G - E(R_i)$ such that $E(H) \neq \emptyset$ and choose a vertex x_{i+1} of $V(H) \cap V(R_i)$. Construct a tour R_i' in H by taking a maximal trail starting at x_{i+1}. Let R_{i+1} be the tour obtained by first traversing R_i starting and ending at x_{i+1}, then traversing R_i' starting and ending at x_{i+1}. Now iterate.

- If R_i is an Euler tour of G then STOP and output R_i.

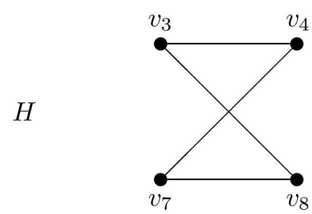
7.6 Example

Consider the graph G shown below.



First Iteration We take $x_1 = v_3$ and construct the tour $R_1 = v_2v_3v_6v_7v_2v_1v_6v_5v_2$ in G.

Second Iteration $G - E(R_1)$ has exactly one component H with $E(H) \neq \emptyset$.



We have $V(R_1) \cap V(H) = \{v_3, v_7\}$. We take $x_2 = v_3$ and construct the tour $R_1' = v_3v_4v_7v_8v_3$ in H. We now 'rotate' R_1 so that it starts and ends at v_3 and put $R_2 = v_3v_6v_7v_2v_1v_6v_5v_2v_3v_4v_7v_8v_3$. Then R_2 is the required Euler tour of G.

7.7 Remark

Since Algorithm 7.5 considers each edge of G exactly once, the time it takes to construct an Euler tour of G is $O(|E(G)|)$.

7.8 Remark

It is straightforward to modify Theorem 7.3 to characterise when a digraph has a directed Euler tour.