5 Matchings

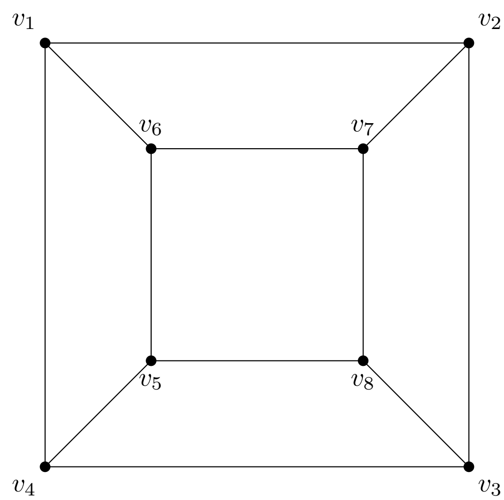
5.1 Matchings in general graphs

5.1.1 Definitions

Let G be a graph and $M \subseteq E(G)$. Then M is a *matching* in G if no two edges of M have a common end-vertex. We say that M is a *maximum matching* if it has maximum cardinality over all matchings in G. A vertex $v \in V(G)$ is *M-saturated* if v is incident with an edge of M. We say that M is a *perfect matching* in G if every vertex of G is M-saturated. Thus, if M is a perfect matching, then $|M| = \frac{1}{2}|V(G)|$ and M is necessarily a maximum matching. Let $match(G)$ denote the size of a maximum matching in G.

5.1.2 Example

$G =$



$M_1 = \{v_1v_2, v_5v_6, v_3v_8\}$ is a matching in G but is not maximum, $M_2 = \{v_1v_2, v_6v_7, v_5v_8, v_4v_3\}$ is a perfect matching and hence is also maximum. Thus $match(G) = 4$. Vertex v_4 is M_1-unsaturated but is M_2-saturated.

5.1.3 Problem

Given a graph G, construct a maximum matching in G.

5.1.4 Definitions

Let G be a graph and $U \subseteq V(G)$. We say that U is a *cover* of G if every edge of G is incident with a vertex in U. We say that U is a *minimum cover* if it has minimum cardinality over all covers of G. Let $cov(G)$ denote the size of a minimum cover of G.

5.1.5 Example

In the graph G of Example 5.1.2, $U_1 = \{v_1, v_2, v_4, v_5, v_7, v_8\}$ and $U_2 = \{v_1, v_3, v_5, v_7\}$ are both covers of G. We have $cov(G) \leq |U_2| = 4$.

5.1.6 Lemma

Let G be a graph. Then $match(G) \leq cov(G)$.

Proof Let M be a maximum matching in G, and U be a minimum cover of G. Since U is a cover of G each edge of M is incident with a vertex in U. Since M is a matching, no two edges in M have a common end vertex. Thus we need at least $|M|$ vertices to cover all edges of M. Hence $match(G) = |M| \leq |U| = cov(G)$.

Lemma 5.1.6 has the following immediate corolly.

5.1.7 Corollary

Let G be a graph. If G has a matching M and a cover U such that $|M| = |U|$, then M is a maximum matching in G and U is a minimum cover of G.

Proof We have

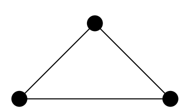
$$match(G) \geq |M| = |U| \geq cov(G). \tag{7}$$

On the other hand, Lemma 5.1.6 implies that $match(G) \leq cov(G)$. Thus equality must hold throughout (7). Hence $match(G) = |M|$ and $cov(G) = |U|$.

Since we have $|M_2| = 4 = |U_2|$ in the graph G of Examples 5.1.2 and 5.1.5, Corollary 5.1.7 implies that $|M_2|$ is a maximum matching in G and $|U_2|$ is a minimum cover of G.

5.1.8 Note

It is not true that we must have equality in Lemma 5.1.6 for all graphs. Consider the following graph G.



It is easy to check that $match(G) = 1$ and $cov(G) = 2$.

5.1.9 Definitions

Let M be a matching in a graph G. An M-*alternating path* in G is a path whose edges alternate between M and $E(G) - M$. An M-*augmenting path* in G is an M-alternating path whose end vertices are M-unsaturated. Thus in Example 5.1.2, $P = v_4v_1v_2v_7$ is an M_1-augmenting path in G. We shall see that M-alternating paths play a similar role to that of f-unsaturated paths in Chapter 3.

5.1.10 Notation

Given two sets A and B, let $A \triangle B = (A - B) \cup (B - A)$ denote the *symmetric difference* between A and B.

5.1.11 Lemma

Let M be a matching in a graph G. Suppose G has an M-*augmenting path* P. Then G has a matching M' with $|M'| = |M| + 1$.

Proof Let $M' = M \triangle E(P)$. Since the ends of P are M-unsaturated, M' is a matching in G and $|M'| = |M| + 1$.

5.1.12 Theorem (J. Petersen, 1891)

Let M be a matching in a graph G. Then M is a maximum matching in G if and only if G has no M-augmenting path.

Proof (a) **Necessity** Suppose M is a maximum matching in G. Then Lemma 5.1.11 implies that G has no M-augmenting path.
(b) **Sufficiency** Suppose M is not a maximum matching and let M' be a matching in G with $|M'| > |M|$. Let $S = M \triangle M'$ and let H be the spanning subgraph of G with $E(H) = S$. Since each vertex of G is incident with at most two edges of S we have $d_H(v) \leq 2$ for all $v \in V(H)$. Thus each component of H is either a path or a cycle. Furthermore, since $d_H(v) = 2$ if and only if v is incident with an edge of M and an edge of M', it follows that the edges in the paths and cycles of H alternate between M and M'. In particular, we deduce that each cycle of H has an even length. Since $|M'| > |M|$, some component of H must be a path which starts and ends with an edge of M'. This path will be the required M-augmenting path in G.

5.1.13 Remark

There is a polynomial algorithm, due to J. Edmonds (1965), which constructs a maximum matching in a graph by searching for alternating paths. Unfortunately, his algorithm is beyond the scope of this course. Instead, we will describe a simpler algorithm which constructs maximum matchings in a special family of graphs.

5.2 Matchings in Bipartite Graphs

5.2.1 Definition

A graph G is *bipartite* with *bipartition* $\{X, Y\}$ if $\{X, Y\}$ is a partition of $V(G)$ and all edges of G join vertices of X to vertices of Y.

5.2.2 Lemma

A graph G is bipartite if and only if G contains no cycles of odd length.

Proof (a) **Necessity** Assume G is bipartite and let $\{X, Y\}$ be a bipartition of G. Suppose G contains a cycle $C = v_1 v_2 \dots v_{2m+1}$ of odd length. Then without loss of generality, $v_1 \in X$. This implies that $v_2 \in Y$, $v_3 \in X$, and so on. Thus $v_{2m+1} \in X$. This is impossible since $v_1 v_{2m+1}$ would be an edge of G incident to two vertices of X.

(b) **Sufficiency** Assume G contains no cycles of odd length. Let H be a component of G, v_0 be a vertex of H, and T be a spanning tree of H rooted at v_0. Let $X = \{v \in V(H) : dist_T(v_0, v) \text{ is even}\}$ and $Y = \{v \in V(H) : dist_T(v_1, v) \text{ is odd}\}$. We will show that $\{X, Y\}$ is a bipartition of H. Suppose not. Without loss of generality there is an edge $x_1 x_2$ in H with $x_1 x_2 \in X$. Let P_1 be the path in T from v_0 to x_1 and P_2 be the path in T from v_0 to x_2. Let $v_0 v_1 \dots v_m$ be the path which is common to both P_1 and P_2, $P_1[v_m, x_1]$ be the segment of P_1 from v_m to x_1, and $P_2[x_2, v_m]$ be the segment of P_2 from x_2 to v_m. Since P_1 and P_2 both have even length, $P_1[v_m, x_1]x_1 x_2 P_2[x_2, v_m]$ is a cycle in H of odd length. This is impossible. Hence $\{X, Y\}$ is a bipartition of H. Thus all components of G are bipartite. This implies that G is bipartite.

5.2.3 Theorem (D. König, 1931)

Let G be a bipartite graph. Then $match(G) = cov(G)$.

Proof Let $\{X, Y\}$ be a bipartition of G and let M be a maximum matching in G. By Lemma 5.1.6, we have $|M| = match(G) \leq cov(G)$. Thus it suffices to show that G has a cover U with $|U| = |M|$.

Let X_0 be the set all M-unsaturated vertices in X and let W be the set of all vertices of G which can be reached by M-alternating paths starting at X_0. Let $X_1 = X \cap W$, $Y_1 = Y \cap W$, $X_2 = X - W$ and $Y_1 = Y \cap W$. Put $U = X_2 \cup Y_1$. We will show that U is a cover of G with $|U| = |M|$.

We first show that U is a cover of G. Suppose not. Let xy be an edge of G which is not covered by U. Then $x \in X_1$ and $y \in Y_2$. Since $x \in X_1 = X \cap W$, there is an M-alternating path $P = x_0 y_1 x_1 y_2 x_2 \dots x_m y_m x$ in G from a vertex $x_0 \in X_0$ to x. Since x_0 is M-unsaturated, $x_0 y_1 \notin M$. Since P is M-alternating, we must have $\{y_1 x_1, y_2 x_2, \dots, y_m x\} \subseteq M$. Now $P' = x_0 y_1 x_1 y_2 x_2 \dots x_m y_m x y$ is an M-alternating path from x_0 to y. This contradicts the fact that $y \in Y_2 = Y - W$.

We next show that every vertex in U is M-saturated. Since $X_2 = X - X_1 \subseteq X - X_0$ and X_0 is the set of all M-unsaturated vertices in X, all vertices in X_2 are M-saturated. If some vertex $y \in Y_1$ was M-unsaturated then the M-alternating path from a vertex $x_0 \in X_0$ to y would be M-augmenting. This would contradict the fact that M is a maximum matching in G by Lemma 5.1.11. Thus all vertices in $U = X_2 \cup Y_1$ are M-saturated.

We next show that every edge in M is incident with a unique vertex of U. Since U is a cover of G, every edge in M is incident with at least one vertex of U. Suppose some edge $xy \in M$ is incident with two vertices of U.

Then $x \in X_2$ and $y \in Y_1$. Since $y \in Y_1 = Y \cap W$, there is an M-alternating path $Q = x_0y_1x_1y_2x_2\dots y_{m-1}x_my$ in G from a vertex $x_0 \in X_0$ to y. Since x_0 is M-unsaturated, $x_0y_1 \notin M$. Since Q is M-alternating, we must have $\{y_1x_1, y_2x_2, \dots, y_{m-1}x_m\} \subseteq M$. Now $Q' = x_0y_1x_1y_2x_2\dots y_{m-1}x_myx$ is an M-alternating path from x_0 to x. This contradicts the fact that $x \in X_2 = X - W$.

We have shown that all edges of M are incident with a unique vertex of U and all vertices of U are incident with a unique edge of M. Thus $|U| = |M|$ and $match(G) = cov(G)$.

The above proof of König's theorem gives rise to an algorithm for finding a maximum matching and a minimum cover in a bipartite graph G. We start with a given matching, and then iteratively increase the size of the matching using augmenting paths. When we find a matching M for which their are no augmenting paths we construct a cover U of G with $|U| = |M|$ as described in the above proof. We search for augmenting paths using the following concept.

5.2.4 Definition

Let G be a bipartite graph with bipartition $\{X, Y\}$. Let M be a matching in G and X_0 be the set of M-unsaturated vertices in X. An *M-alternating forest in G rooted at X_0* is a forest F such that $X_0 \subseteq V(F)$, each component of F contains a unique vertex of X_0, and each path in F is M-alternating. We say that F is *maximal* if it is not contained in a larger M-alternating forest rooted at X_0.

5.2.5 König's algorithm

We are given a bipartite graph G and a bipartition $\{X, Y\}$ of G. We construct a maximum matching M in G and a minimum cover U of G (with $|M| = |U|$).

Initial Step We construct a matching M_1 in G by 'greedily' choosing edges with no common end vertices until we cannot choose any more.

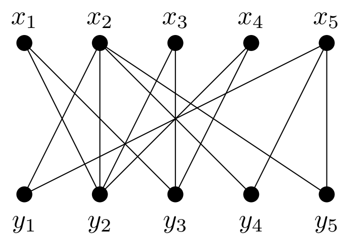
Iterative Step Suppose we have constructed a matching M_i in G for some $i \geq 1$. Let X_0 be the set of M_i-unsaturated vertices in X. Grow a maximal M_i-alternating forest rooted at X_0, for example by depth first search.

- If some component T of F contains an M_i-unsaturated vertex other than the root, then the unique path P in T from the root to this vertex is an M_i-augmenting path. Put $M_{i+1} = M_i \triangle E(P)$ and iterate.

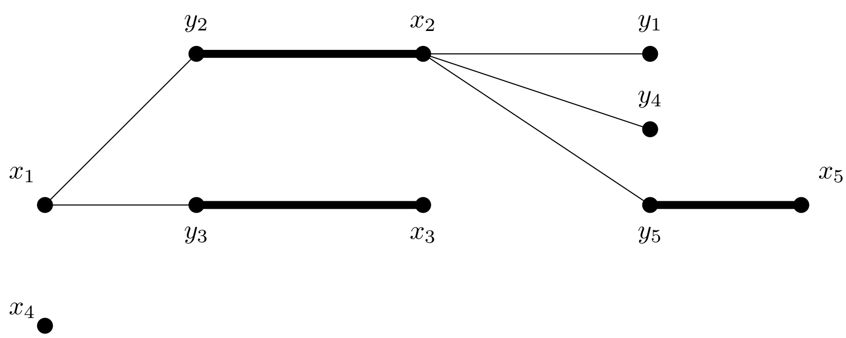
- If every component of F contains exactly one M_i-unsaturated vertex then STOP. Put $M = M_i$ and $U = [X - V(F)] \cup [Y \cap V(F)]$ and output M and U.

5.2.6 Example

Let G be the bipartite graph shown below.



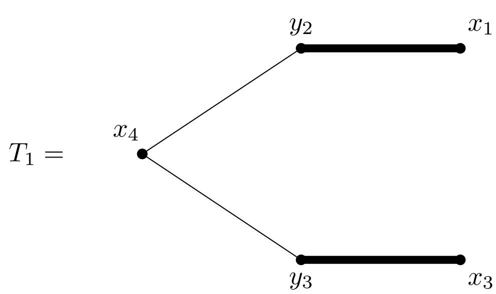
Let $M_1 = \{x_2y_2, x_3y_3, x_5y_5\}$.
First iteration Grow an M_1-alternating forest F_1 rooted at the M_1-unsaturated vertices x_1, x_4.



The component of F_1 which contains x_1 contains another M_0-unsaturated vertex y_1. Let $P_1 = x_1y_2x_2y_1$ and put

$$M_2 = M_1 \triangle E(P_1) = \{x_2y_1, x_1y_2, x_3y_3, x_5y_5\}.$$

Second iteration Grow an M_2-alternating forest F_2 rooted at the M_2-unsaturated vertex x_4.



No component of F_2 contains an M_2-unsaturated vertex other than its root. Thus M_2 is a maximum matching in G. Put $U = (V(F_2) - X) \cup (V(F_2) \cap X) =$

$\{x_2, x_5, y_2, y_3\}$. Then U is a minimum cover of G. We have $|M_2| = 4 = |U|$.

5.2.7 Lemma

The time taken for König's algorithm to construct a maximal matching and a minimum cover in a graph G is $O(|V(G)| \times |E(G)|)$.

Proof The time taken to grow an alternating forest in each iteration of the algorithm is $O(|E(G)|)$. (To see this we suppose that we grow the forest F using breadth first search and that the vertices of F are added in the order $u_1, u_2, \ldots, u_t$. Then we first consider all edges incident to u_1, then all edges incident to u_2, and so on. Thus the time taken is $O(\sum_{u \in V(G)} d_G(u)) = O(2|E(G)|) = O(|E(G)|)$.) Since each iteration increases the size of the matching, the number of iterations is at most $\lfloor |V(G)|/2 \rfloor$. Hence the total running time of the algorithm is $O(|V(G)| \times |E(G)|)$.

5.2.8 Remark

As noted in Remark 5.1.13, there is a polynomial algorithm due to Edmonds (1965) which constructs a maximum matching in a graph which is not necessarily bipartite. There is no known polynomial algorithm, however, for finding a minimum cover in a graph which is not bipartite.

Matchings which saturate one side of the bipartition

Suppose G is a bipartite graph with bipartition $\{X, Y\}$ where $|X| \le |Y|$. Since X is a cover of G, every matching in G has size at most $|X|$, and every matching of size $|X|$ will saturate every vertex of X. We can use Konig's Therem to deduce a simple characterization of when G has such a matching. We need the following concept.

5.2.9 Definition

Let G be a graph and $S \subseteq V(G)$. Then the *neighbour set* of S, $\Gamma_G(S)$, is the set of all vertices of G which are adjacent to at least one vertex of S.

5.2.10 Theorem (P. Hall, 1935)

Let G be a bipartite graph with bipartition $\{X, Y\}$. Then exactly one of the following alternatives hold.
(a) G has a matching which saturates every vertex of X.
(b) There exists a set $S \subseteq X$ such that $|\Gamma_G(S)| < |S|$.

Proof Suppose G has a matching M which saturates X. Choose $S \subseteq X$. Since each vertex of S is matched by an edge of M to a distinct vertex of Y, we must have $|\Gamma_G(S)| \ge |S|$. Hence (b) cannot occur.

Suppose G does not have a matching which saturates X. Then $match(G) < |X|$. Since $match(G) = cov(G)$ by König's Theorem, we have $cov(G) < |X|$. Let U be a minimum cover of G and let $S = X - U$ and $T = Y \cap U$. Since U is a cover of G, there are no edges in G from S to $Y - T$. Thus $\Gamma_G(S) \subseteq T$. We have

$$|X| > cov(G) = |U| = |U \cap X| + |U \cap Y| = |X - S| + |T| = |X| - |S| + |T|.$$

Thus $|S| > |T| \geq |\Gamma_G(S)|$, and (b) holds.

5.2.11 Note

The above proof is constructive. If G does not have a matching which saturates X, then we can find a set $S \subseteq X$ with $|\Gamma_G(S)| < |S|$ by constructing a minimum cover U of G and putting $S = X - U$.

5.2.12 Example

The graph G of Example 5.2.6 has $match(G) = 4 < |X|$ and hence does not have a matching which saturates X. We have sen that $U = \{x_2, x_5, y_2, y_3\}$ is a minimum cover of G. Putting $S = X - U = \{x_1, x_3, x_4\}$ we have $\Gamma_G(S) = \{y_2, y_3\}$ and $|\Gamma_G(S)| = 2 < 3 = |S|$.