2 Bridges and Trees

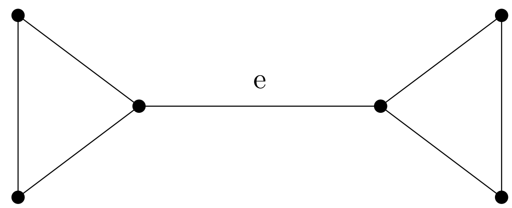
2.1 Bridges

2.1.1 Definition

Let e be an edge of a graph G. Then e is a *bridge* of G if $G - e$ has more connected components than G. Thus, if G is connected, then e is a bridge of G if $G - e$ is disconnected.

Example

G =



2.1.2 Lemma

Let e be a bridge of a connected graph G and let u and v be the end vertices of e. Then $G - e$ has exactly two components H_1 and H_2 with $u \in V(H_1)$ and $v \in V(H_2)$.

Proof Let $H = G - e$. Since e is a bridge of G, H has at least two connected components. We shall prove that there are exactly two components by showing that every vertex of H is joined to either u or v by a path in H. Choose $w \in V(H)$. Since G is connected, w is joined to u by a path $P = v_0 e_1 v_1 e_2 v_2 \ldots v_{m-1} e_m v_m$ in G (where $w = v_0$ and $u = v_m$). If $e \notin E(P)$ then P is a path joining w to u in H. On the other hand, if $e \in E(P)$ then we must have $e = e_m$ and hence $v_{m-1} = v$. Thus $P' = v_0 e_1 v_1 e_2 v_2 \ldots v_{m-1}$ is a path from w to v in H.

In both cases we deduce that w belongs to the same connected component of H as either u or v. Since this holds for all vertices w of H, it follows that H has exactly two components, each containing either u or v.

2.1.3 Corollary

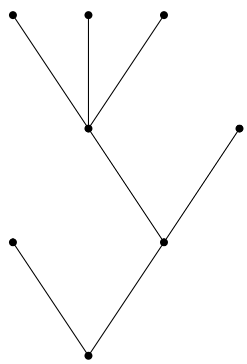
Let e be an edge of a graph G. Then e is a bridge if and only if e is not contained in any cycle of G.

Proof Let the end vertices of e be u and v. We may assume that G is connected since otherwise we may just consider the connected component containing e. Using Lemma 2.1.2, it follows that e is a bridge if and only if there is no path in $G-e$ joining u and v. On the other hand, e is contained in a cycle C of G if and only if $C-e$ is a path in $G-e$ joining u and v. The corollary now follows.

2.2 Trees

2.2.1 Definition

A *tree* is a connected graph which contains no cycles.
Example



2.2.2 Lemma

A connected graph G is a tree if and only if every edge of G is a bridge.

Proof Immediate from Corollary 2.1.3.

2.2.3 Lemma

Let T be a tree. Then $|E(T)| = |V(T)| - 1$.

Proof
We use induction on $|V(T)|$.
Base Case $|V(T)| = 1$. Since T has no cycles, it has no loops and hence $|E(G)| = 0$. Thus $|E(T)| = 0 = |V(T)| - 1$.
Induction Hypothesis Suppose that $k \geq 2$ is an integer and that the lemma is true for all trees with at most $k-1$ vertices.
Inductive Step Let T be a tree with k vertices. Let e be an edge of T. By

Lemma 2.2.2, e is a bridge of T and hence by Lemma 2.1.2, $T-e$ has exactly two connected components T_1 and T_2. Since T_1 and T_2 are subgraphs of a tree, they have no cycles. Thus T_1 and T_2 are trees. Furthermore they each have fewer edges that T so we can apply induction to T_1 and T_2 to give $|E(T_1)| = |V(T_1)| - 1$ and $|E(T_2)| = |V(T_2)| - 1$. Thus

$$\begin{aligned}
|E(T)| &= |E(T-e)| + 1 \\
&= |E(T_1)| + |E(T_2)| + 1 \\
&= (|V(T_1)| - 1) + (|V(T_2)| - 1) + 1 \\
&= |V(T_1)| + |V(T_2)| - 1 \\
&= |V(T)| - 1.
\end{aligned}$$

2.2.4 Lemma

Let T be a tree with at least two vertices. Then T has at least two vertices of degree one.

Proof Let $P = v_0e_1v_1 \ldots e_mv_m$ be a path of maximum length in D. We show that e_0 is the only edge of T incident with v_0. If e were an edge in G joining v_0 to a vertex $x \in V(T) - V(P)$ then $P' = xev_0e_1v_1 \ldots e_mv_m$ would be a longer path than P. If $e \neq e_0$ were an edge in G joining v_0 to a vertex $v_i \in V(P)$ then $C = v_0e_1v_1 \ldots e_iv_iev_0$ would be cycle in T, which is impossible since T is a tree. Thus e_0 is the only edge of T incident with v_0 and $d_T(v_0) = 1$. Similarly, $d_D^+(v_m) = 0$.