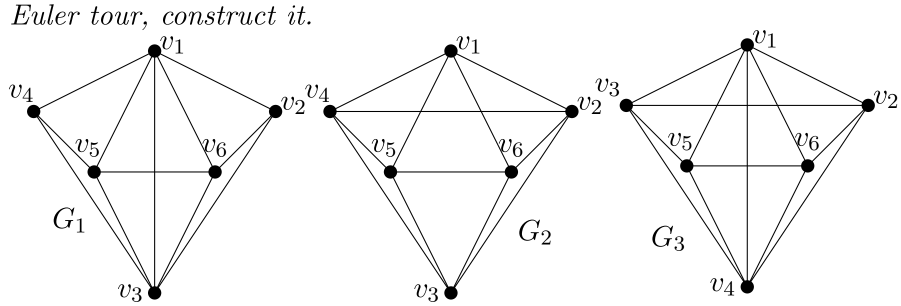
MAS210 Graph Theory Exercises 9 Solutions

Q1 *Determine whether each of the following graphs G_1, G_2, G_3 have an Euler tour or an Euler trail. In the case(s) when there is an Euler trail but no Euler tour, construct it.*



G_1 has neither an Euler tour or an Euler trail since it has four vertices of odd degree.
G_2 has an Euler tour since it has no vertices of odd degree.
G_3 has an Euler trail but no Euler tour since it has two vertices of odd degree.
An Euler trail in G_3 is $R = v_1v_2v_4v_3v_2v_6v_5v_3v_1v_5v_4v_6v_1v_4$. (Solution not unique.)

Q2 *Prove that every connected graph has a closed walk which uses every edge exactly twice. Which graphs have a closed walk which uses every edge exactly three times? Justify your answer.*

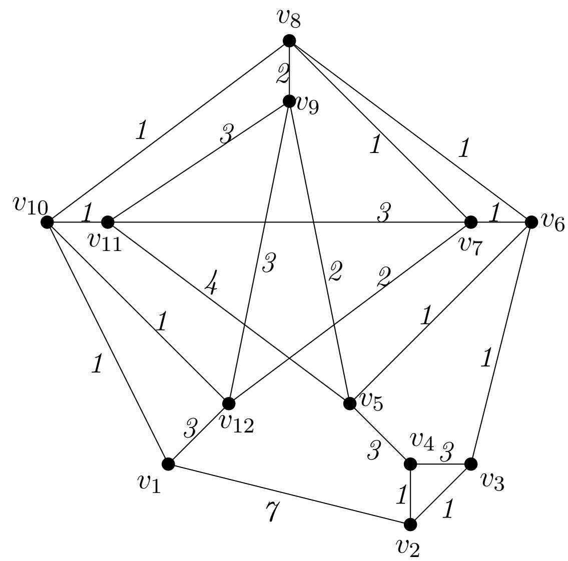
Let G be a connected graph. Construct a new graph H from G by 'doubling' each edge of G i.e. replacing each single edge e of G by a multiple edge consisiting of two edges with the same end vertices as e. For each $v \in V(G)$ we have $d_H(v) = 2d_G(v)$. Thus each vertex of H has even degree. Hence H has an Euler tour R. Then R corresponds to a closed walk in G which uses each edge exactly twice.

Let G be a graph (without any vertices of degree zero). Then G has a closed walk which uses every edge exactly three times if and only if G is connected and each vertex of G has even degree.

Proof G has a closed walk W which uses every Construct a new graph H from G by 'tripling' each edge of G i.e. replacing each single edge e of G by a multiple edge consisiting of three edges with the same end vertices as e.

Then G has a closed walk which uses every edge exactly three times if and only if H has an Euler tour. Since 'tripling' each edge of G will not change the connectedness of G and will not change the parity of the degrees of the vertices of G, this occurs if and only if G is connected and each vertex of G has even degree.

Q3 *Let N be the network shown below.*

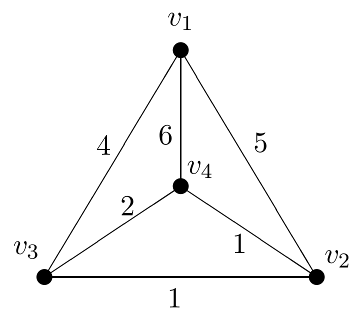


Let W_1 and W_2 be shortest walks in N which traverse every arc of N at least once, and are such that:

- *W_1 is closed;*
- *W_2 starts at v_2 and ends at v_3.*

Suppose the length of W_i is $w(N) + m_i$ for each $i \in \{1, 2\}$. Determine m_1 and m_2 giving a short explanation of how you obtain each answer.

(a) To construct W_1, we need to construct a minimum weight Eulerian extension of N. Since the set of vertices of odd degree in N is $U = \{v_1, v_2, v_3, v_4\}$, we construct a weighted complete graph with vertex set U, in which the weight of the edge v_iv_j is given by the length of a shortest path in N joining v_i and v_j. This gives us the following network N_{odd}.



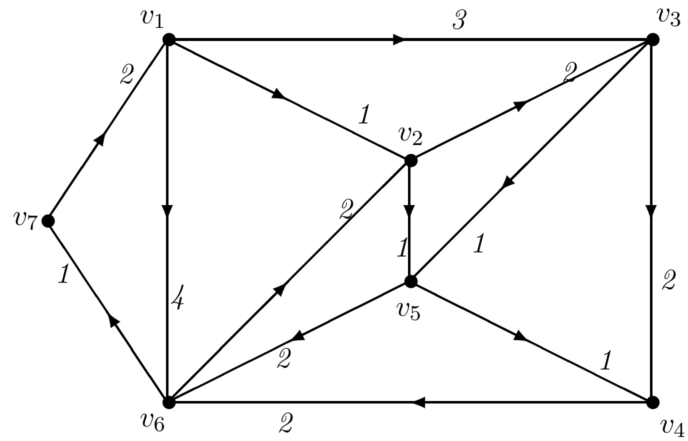
The minimum weight perfect matching in N_{odd} is $M = \{v_1v_3, v_2v_4\}$. Since $w(M) = 4 + 1 = 5$, we have $m_1 = 5$.

(b) To construct W_2, we need to construct a minimum weight extension of N in which v_2, v_3 have odd degree and all other vertices of have even degree. Since the set of vertices of odd degree in N is $U = \{v_1, v_2, v_3, v_4\}$, we can accomplish this by 'doubling' edges along a shortest path from v_1 to v_4 in N. The shortest v_1v_4-path in N, $P = v_1v_{10}v_8v_6v_3v_2v_4$, has length six. Thus $m_2 = 6$.

Q4 *State, without proving, a necessary and sufficient condition for a digraph to have a directed Euler tour.*

Let D be a digraph. Then D has a directed Euler tour if and only if D is strongly connected and each vertex v of G has $d^+(v) = d^-(v)$.

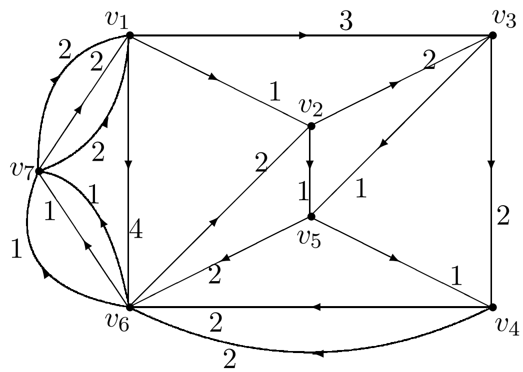
Q5 *Let N be the directed network shown below.*



Let W be a shortest closed directed walk in N which traverses every arc of N at least once. Suppose that the length of W is $w(N) + m$. Determine m giving a short explanation of how you obtain your answer. Describe briefly how the walk W could be constructed.

To construct W, we need to construct a minimum weight Eulerian extension of N. The set of vertices of N whose in-degree and out-degree are not the same is $U = \{v_1, v_4, v_6\}$. We have $d^+(v_1) - d^-(v_1) = 3 - 1 = 2$, $d^-(v_4) - d^+(v_4) = 2 - 1 = 1$, and $d^-(v_6) - d^+(v_6) = 3 - 2 = 1$. Thus we can construct a minimum weight Eulerian extension of N by 'doubling' each arc of N which belongs to a shortest directed path from v_4 to v_1 in N, and a shortest directed path from v_6 to v_1 in N. The shortest directed v_4v_1-path in N, $P_1 = v_4v_6v_7v_1$, has length five. The shortest directed v_6v_1-path in N, $P_2 = v_6v_7v_1$, has length three. Thus $m = 5 + 3 = 8$.

We construct the walk W by first constructing the Eulerian extension of N, by 'doubling' arcs along P_1 and P_2. This gives the directed network N^* shown below.



We then construct a directed Euler tour in N^*.