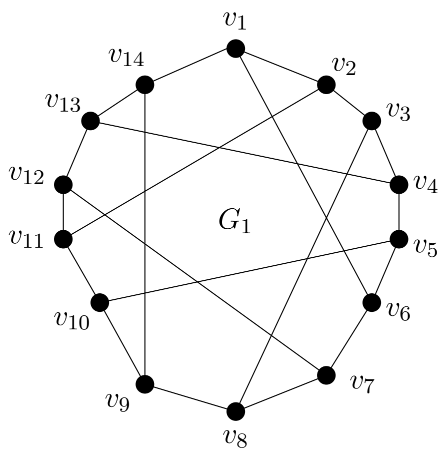
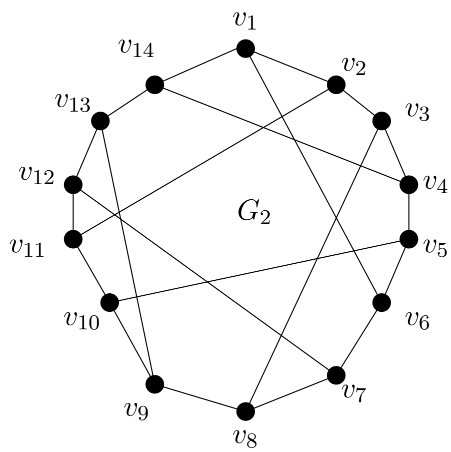
MAS210 Graph Theory Exercises 7 Solutions

Q1 *Determine whether each of the following graphs G_1 and G_2 are bipartite. Justify your answers.*





G_1 is bipartite since $X = \{v_1, v_3, v_5, v_7, v_7, v_{11}, v_{13}\}$ and $Y = \{v_2, v_4, v_6, v_8, v_{10}, v_{12}, v_{14}\}$ is a bipartition of G. (It is easy to check that every edge of G has one end in X and one end in Y.

G_2 is not bipartite since it contains the cycle $C = v_1v_2v_3v_4v_{14}v_1$ which has odd length. (Or give a proof along the lines that if G has bipartition $\{X, Y\}$ then we may assume $v_1 \in X$. This implies that: $v_2, v_6, v_{14} \in Y$; $v_3, v_{11} \in X$; $v_4, v_8 \in Y$. Thus v_4v_{14} is an edge joining two vertices of Y. Contradiction.)

Q2 (a) *Prove that if a graph G contains a cycle of odd length then G is not bipartite.*

(b) Suppose G is a connected graph which contains no cycles of odd length. Choose $v_0 \in V(G)$ and let T be a spanning tree of G rooted at v_0. Let $X = \{v \in V(G) : dist_T(v_0, v)$ is even$\}$ and $Y = \{v \in V(G) : dist_T(v_0, v)$ is odd$\}$. Prove that G is bipartite with bipartition $\{X, Y\}$.

(c) Deduce that a graph is bipartite if and only if it contains no cycles of odd length.

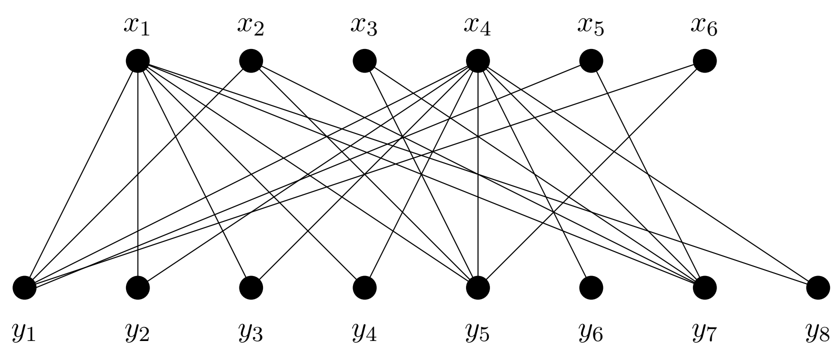
(a) Let $C = v_1v_2 \ldots v_{2t}v_{2t+1}v_1$ be a cycle of odd length in G. Suppose G has bipartition $\{X, Y\}$. Then we may assume $v_1 \in X$. This implies that:

$v_2 \in Y$; $v_3 \in X$; $v_4 \in Y$ and so on. Thus $\{v_1, v_3, v_5, \ldots, v_{2t+1}\} \subseteq X$ and $\{v_2, v_4, v_6, \ldots, v_{2t}\} \subseteq Y$. But then $v_1 v_{2t+1}$ is an edge joining two vertices of X. Contradiction.

(b) Suppose that $\{X, Y\}$ is not a bipartition of G. Without loss of generality there is an edge $x_1 x_2$ in G with $x_1 x_2 \in X$. Note that $x_1 x_2 \notin E(T)$ since, if it were, then we would have $dist_T(v_0, x_1) = dist_T(v_0, x_2) \pm 1$, which would contradict the fact that $dist_T(v_0, x_1)$ and $dist_T(v_0, x_2)$ are both even. Let P_1 be the path in T from v_0 to x_1 and P_2 be the path in T from v_0 to x_2. Let $v_0 v_1 \ldots v_m$ be the path which is common to both P_1 and P_2, $P_1[v_m, x_1]$ be the segment of P_1 from v_m to x_1, and $P_2[x_2, v_m]$ be the segment of P_2 from x_2 to v_m. Since P_1 and P_2 both have even length, $P_1[v_m, x_1]x_1 x_2 P_2[x_2, v_m]$ is a cycle in H of odd length. This is impossible. Hence $\{X, Y\}$ is a bipartition of G.

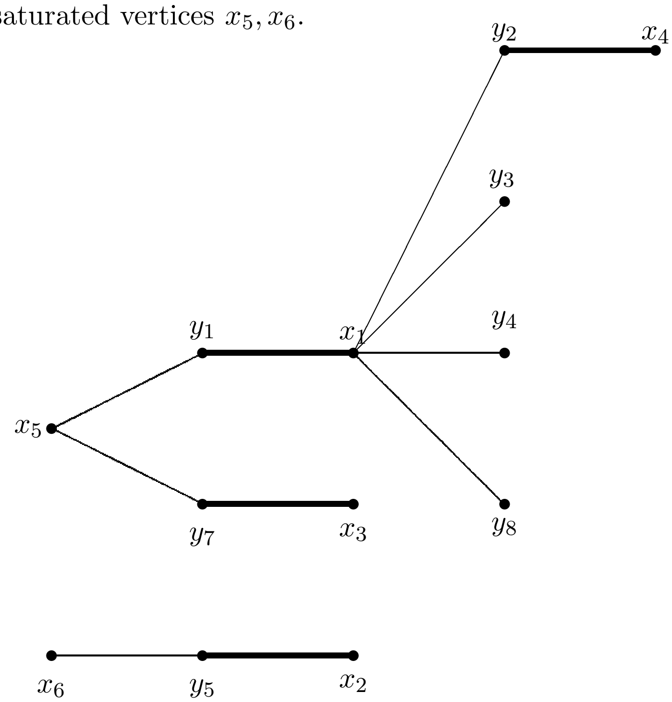
(c) Suppose G is bipartite. Then G has no cycles of odd length by (a).
Suppose G has no cycles of odd length. Then each connected component of G is bipartite by (b). Thus G is bipartite.

Q3 *Use König's algorithm to construct a maximum matching and a minimum cover in the following bipartite graph, starting with the matching* $M_1 = \{x_1 y_1, x_2 y_5, x_3 y_7, x_4 y_2\}$.



Justify the facts that your matching is maximum and your cover is minimum.

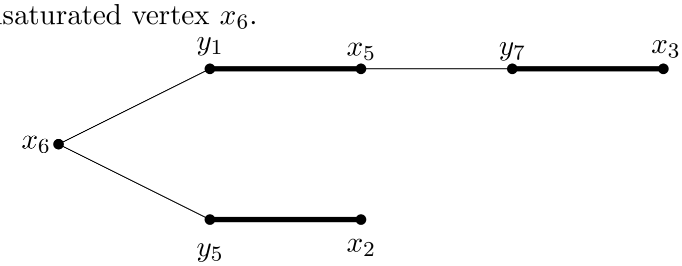
First Iteration Grow a maximal M_1-alternating forest F_1 rooted at the M_1-unsaturated vertices x_5, x_6.



The forest contains the M_1-augmenting path $P_1 = x_5y_1x_1y_8$.

Let $M_2 = M_1 \triangle E(P) = \{x_5y_1, x_1y_8, x_2y_5, x_3y_7, x_4y_2\}$.

Second Iteration Grow a maximal M_2-alternating forest F_2 rooted at the M_2-unsaturated vertex x_6.

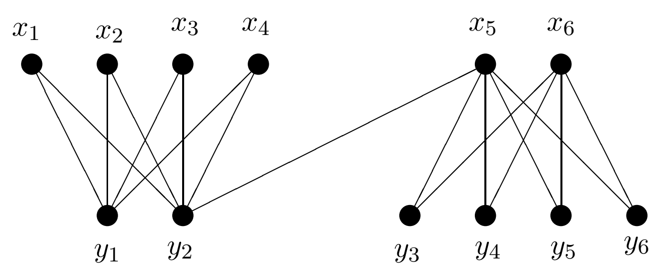


The tree does not contain any other M_2-unsaturated vertices so M_2 is a maximum matching in G. Hence M_2 is a maximum matching in G. Let $U = (X - V(F_2)) \cup (Y \cap V(F_2)) = \{x_1, x_4, y_1, y_5, y_7\}$. Then U is a minimum cover of G.

Justification. We know that, for all matchings M of G, we have $|M| \leq |U| = 5$. Since $|M_2| = 5$, M_2 is a maximum matching in G. Similarly, for all covers U' of G, we have $|U'| \geq |M_2| = 5$. Since $|U| = 5$, U is a minimum cover of G.

Q4 *Use König's theorem to construct a connected bipartite graph G with bipartition $\{X, Y\}$ such that $|X| = 6 = |Y|$, $match(G) = 4$, and $d_G(v) \geq 2$ for all $v \in V(G)$. Justify the fact that your graph G has $match(G) = 4$.*

Let $X = \{x_1, x_2, \ldots, x_6\}$ and $Y = \{y_1, y_2, \ldots, y_6\}$ be the bipartition of G. Let M be a maximum matching in G and U be a minimum cover of G. Then $|M| = 4 = |U|$. Let $X_1 = X - U$, $X_2 = X \cap U$, $Y_1 = Y \cap U$ and $Y_2 = Y - U$. Since U is a cover, there are no edges of G from X_1 to Y_2. Thus all edges incident to X_1 join X_1 to Y_1. Since $d_G(x) \geq 2$ for all $x \in X_1$, we have $|Y_1| \geq 2$. Similarly $|X_2| \geq 2$. Since $U = X_2 \cup Y_1$ and $|U| = 4$ we must have $|X_2| = 2 = |Y_1|$. Without loss of generality, let $X_2 = \{x_5, x_6\}$ and $Y_1 = \{y_1, y_2\}$. Then $X_1 = \{x_1, x_2, x_3, x_4\}$ and $Y_2 = \{y_3, y_4, y_5, y_6\}$. Let G be obtained by adding all edges from X_1 to Y_1 and all edges from X_2 to Y_2. We must also add at least one edge from X_2 to Y_1 to ensure that G is connected. This gives us, for example, the following graph G.



Let $M = \{x_1y_1, x_2y_2, x_5y_5, x_6y_6\}$ and $U = \{x_5, x_6, y_1, y_2\}$. Then M is a matching in G, U is a cover in G and $|M| = 4 = |U|$. Hence M is a maximum matching and U is a minimum cover.