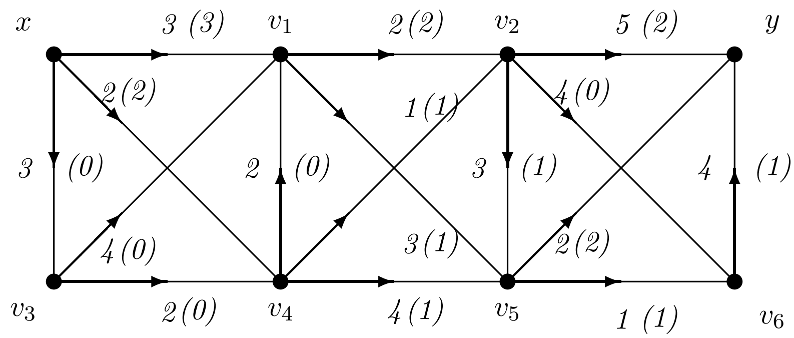
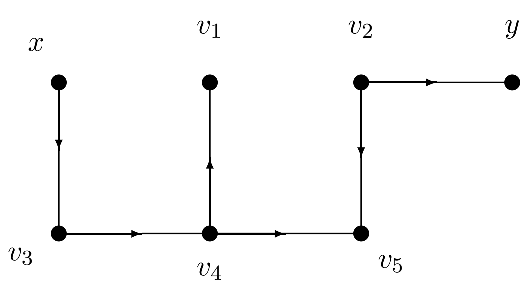
MAS210 Graph Theory Exercises 5 Solutions

Q1 *Consider the following directed network N.*

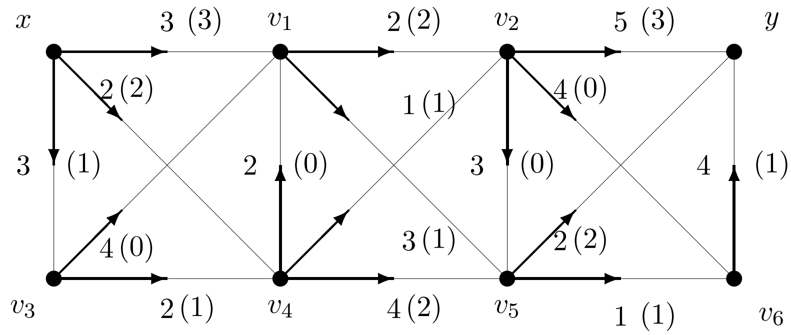


The numbers in brackets define an xy-flow f_1 in N. The numbers not in brackets define the capacities of the arcs of N.
(a) Determine the value of f_1.
(b) Grow a maximal f_1-unsaturated tree T_1 rooted at x and use T_1 to construct an xy-flow f_2 in x with $val(f_2) > val(f_1)$.
(c) Grow a maximal f_2-unsaturated tree T_2 rooted at x and use T_2 to construct a set $U \subset V(N)$ with $x \in U$, $y \in V(N) - U$ and $c^+(U) = val(f_2)$. Explain why this equality implies that f_2 is an xy-flow of maximum value in N and $A_N^+(U)$ is an xy-arc cut of minimum capacity.

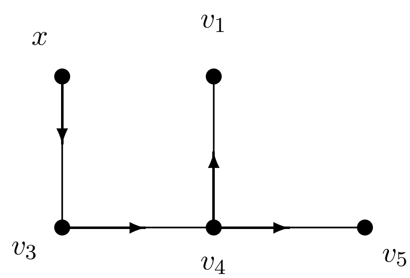
(a) $val(f_1) = 3 + 2 + 0 = 5$.
(b) Maximal f_1-unsaturated tree rooted at x is



Use the f_1-unsaturated path $xv_3v_4v_5v_2y$ to send one extra unit of flow from x to y to give the new xy-flow f_2 shown below.



(c) Maximal f_1-unsaturated tree rooted at x is



The tree does not reach y. Put $U = V(T_2) = \{x, v_3, v_4, v_1, v_5\}$. We have $c^+(U) = 2+1+2+1 = 6 = 1+2+3 = val(f_2)$. We know from lectures that the value of any xy-flow is at most $c^+(U)$. Since $val(f_2) = c^+(U)$, f_2 must be an xy-flow of maximum value. Similarly, we know that the capacity of any xy-arc-cut is at least $val(f_2)$. Since $c^+(U) = val(f_2)$, $A_N^+(U)$ must be an xy-arc-cut of minimum capacity.

Q2 *Let N be a directed network, $x, y \in V(N)$, and let f be an xy-flow in N. Suppose that P is an f-unsaturated path from x to y in N. Define $g : A(N) \to \mathbb{Z}$ by*

$$g(e) = \begin{cases} f(e) & \text{if } e \notin A(P) \\ f(e)+1 & \text{if } e \text{ is a forward arc of } P \\ f(e)-1 & \text{if } e \text{ is a backward arc of } P \end{cases}$$

Prove that g is an xy-flow in N and that $val(g) = val(f) + 1$.

To prove that g is an xy-flow in N, we need to show
(i) $0 \le g(e) \le c(e)$ for all $e \in A(N)$, and
(ii) $g^+(v) = g^-(v)$ for all $v \in V(N) - \{x, y\}$.
Condition (i) follows since $0 \le f(e) \le c(e)$ for all $e \notin A(P)$, $0 \le f(e) \le c(e) - 1$ for all forward arcs e of P, and $1 \le f(e) \le c(e)$ for all backward arcs e of P.

To verify (ii) we choose $v \in V(N) - \{x, y\}$. We need to consider different cases.
Case 1: $v \notin V(P)$. Then $g(e) = f(e)$ for all arcs incident to v so $g^+(v) = f^+(v) = f^-(v) = g^-(v)$.
Case 2: v lies on P. let e_1 be the arc preceding v on P and e_2 be the arc following v on P.
Case 2.1: e_1 and e_2 are both forward arcs of P. Then $g^+(v) = f^+(v) + 1 = f^-(v) + 1 = g^-(v)$.
Case 2.2: e_1 and e_2 are both backward arcs of P. Then $g^+(v) = f^+(v) - 1 = f^-(v) - 1 = g^-(v)$.
Case 2.3: e_1 is a forward arc of P and e_2 is a backward arc of P. Then $g^+(v) = f^+(v) = f^-(v) = g^-(v)$.
Case 2.4: e_1 is a backward arc of P and e_2 is a forward arc of P. Then $g^+(v) = f^+(v) = f^-(v) = g^-(v)$.

We next show that $val(g) = val(f) + 1$. Let e_0 be the arc of P incident to x. We have $g(e) = f(e)$ for all other arcs e of N which are incident to x. If e_0 is a forward arc of P then $g(e_0) = f(e_0) + 1$. Thus $g^+(v) = f^+(v) + 1$, $g^-(v) = f^-(v)$ and $val(g) = val(f)+1$. On the other hand, if e is a backward arc of P then $g(e_0) = f(e_0) - 1$. Thus $g^+(v) = f^+(v)$, $g^-(v) = f^-(v) - 1$ and $val(g) = val(f) + 1$.

Q3 *Let N be a directed network, $x, y \in V(N)$, and let f be an xy-flow in N of maximum value. Let U be the set of all vertices which can be reached from x by f-unsaturated paths. Prove that*
(a) $y \notin U$.
(b) All arcs $e \in A_N^+(U)$ satisfy $f(e) = c(e)$ and all arcs $e \in A_N^-(U)$ satisfy $f(e) = 0$.
(c) $val(f) = c^+(U)$.

(a) If $y \in U$ then N would have an f-unsaturated path from x to y. This would contradict the fact that f is an xy-flow of maximum value (by Q2 above). Thus $y \notin U$.

(b) Choose $e \in A_N^+(U)$. Then $e = uv$ for some vertices $u \in U$ and $v \notin U$. Since $u \in U$ there is an f-unsaturated path P from x to u in N. If $f(e) < c(e)$, then we could add e to P to get an f-unsaturated path P from x to v in N. This would contradict the fact that $v \notin U$. Hence $f(e) = c(e)$.
Choose $e \in A_N^-(U)$. Then $e = vu$ for some vertices $u \in U$ and $v \notin U$. Since $u \in U$ there is an f-unsaturated path P from x to u in N. If $f(e) > 0$, then we could add e to P to get an f-unsaturated path P from x to v in N. This would contradict the fact that $v \notin U$. Hence $f(e) = 0$.

(c) Using (b) we have

$$val(f) = f^+(U) - f^-(U) = \sum_{e \in A_N^+(U)} f(e) - \sum_{e \in A_N^-(U)} f(e) = \sum_{e \in A_N^+(U)} c(e) - 0 = c^+(U).$$

Q4 *Let D be a digraph and $x, y \in V(D)$. Suppose that $d_D^+(v) = d_D^-(v)$ for all $v \in V(D) - \{x, y\}$. Let $U \subset V(D)$ with $x \in U$ and $y \in V(D) - U$. Let $d_D^+(U)$ be the number of arcs of D from U to $V(D) - U$ and $d_D^-(U)$ be the number of arcs of D from $V(D) - U$ to U.*
(a) Prove that $d_D^+(U) - d_D^-(U) = d_D^+(x) - d_D^-(x)$.
(b) Deduce that if $d_D^+(x) > d_D^-(x)$ then there exists a directed path from x to y.

(a) Since $d_D^+(v) = d_D^-(v)$ for all $v \in U - x$, we have

$$d_D^+(x) - d_D^-(x) = \sum_{u \in U} d_D^+(u) - \sum_{u \in U} d_D^-(u).$$

The sums on the right hand side of the above equality count the arcs incident with vertices of U. There are three alternatives for such an arc e.

- e is incident with two vertices of U. Then e is counted once in $\sum_{u \in U} d_D^+(u)$ and once in $\sum_{u \in U} d_D^-(u)$ so its contribution to $\sum_{u \in U} d_D^+(u) - \sum_{u \in U} d_D^-(u)$ is zero.
- $e \in A_N^+(U)$. Then e is counted once in $\sum_{u \in U} d_D^+(u)$ so its contribution to $\sum_{u \in U} d_D^+(u) - \sum_{u \in U} d_D^-(u)$ is 1.
- $e \in A_N^-(U)$. Then e is counted once in $\sum_{u \in U} d_D^-(u)$ so its contribution to $\sum_{u \in U} d_D^+(u) - \sum_{u \in U} d_D^-(u)$ is -1.

Thus

$$\sum_{u \in U} d_D^+(u) - \sum_{u \in U} d_D^-(u) = |A_N^+(U)| - |A_N^-(U)| = d_D^+(U) - d_D^-(U).$$

Hence $d_D^+(x) - d_D^-(x) = d_D^+(U) - d_D^-(U)$.
(b) Let U be the set of all vertices of N which can be reached from x by directed paths. Suppose $y \notin U$. By (a), $d_D^+(U) - d_D^-(U) = d_D^+(x) - d_D^-(x) > 0$. In particular, $d_D^+(U) > 0$. Thus there is an arc e from some vertex $u \in U$ to some vertex $v \in V(D) - U$. But then we can add e to a directed path from x to u to get a directed path from u to v. This contradicts the fact that $v \notin U$. Thus we must have $y \in U$. Hence there is a directed path in D

from x to y.

Q5 *Let N be an acyclic directed network and $\{x_1, x_2, \ldots, x_n\}$ be an acyclic labeling of D.*
(a) Explain why the time that Moravék's algorithm takes to construct the out-arborescence T_i from the out-arborescence T_{i-1} is $O(d_N^-(x_i))$.
(b) Deduce that the total time taken by Moravék's algorithm to construct a spanning out-arborescence of N is $O(|A(N)|)$.

(a) In the i'th iteration of Morávek's algorithm, we consider each arc entering x_i and choose the arc $x_j x_i$ entering x_i for which $dist_{T_{i-1}}(x_1 x_j) + w(x_j x_i)$ is as large as possible. Thus the time taken by the i'th iteration is $O(d_N^-(x_i))$.
(b) Since $\sum_{i=1}^n d_N^-(x_i)) = |A(N)|$, the time taken for the algorithm to construct a spanning out arborescence is $O(\sum_{i=1}^n d_N^-(x_i)) = O(|A(N)|)$.