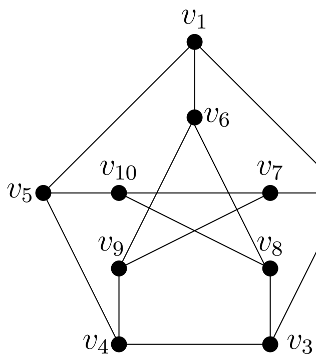
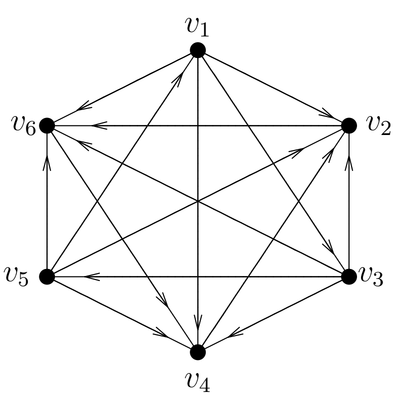
MAS210 Graph Theory Exercises 1 Solutions

Q1 Find examples of each of the following kinds of walks in the graph G below, and give their lengths: (a) a shortest path from v_1 to v_8; (b) a longest path from v_1 to v_8; (c) a shortest cycle in G; (d) a longest cycle in G.

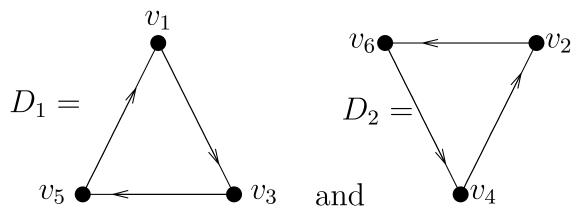


The walks given in the solutions below are not unique (but their lengths are).
(a) A shortest path from v_1 to v_8 is $P_1 = v_1v_6v_8$. Length of P_1 is 2.
(b) A longest path from v_1 to v_8 is $P_2 = v_1v_6v_9v_7v_2v_3v_4v_5v_{10}v_8$. Length of P_1 is 9.
(c) A shortest cycle in G is $C_1 = v_1v_2v_3v_4v_5v_1$. Length of C_1 is 5.
(d) A longest cycle in G is $C_2 = v_1v_2v_3v_4v_5v_{10}v_7v_9v_6v_1$. Length of C_2 is 9.

Q2 Draw the strongly connected components of the following digraph D.



Strongly connected components of D are



Q3 (a) Prove that if a digraph has a closed directed walk of length at least one, then it has a directed cycle.

Proof Let $W = v_1e_1v_2e_2 \ldots v_me_mv_0$ be a closed directed walk in D such that W has length at least one and, subject to this condition, W is as short as possible. Suppose W is not a directed cycle. Then either $e_i = e_j$ or $v_i = v_j$ for some $1 \leq i < j \leq m$. If $e_i = e_j$ then, since e_i has a unique tail, we must also have $v_{i-1} = v_{j-1}$. Thus we may assume that $v_i = v_j$ for some $1 \leq i < j \leq m$. Let $W_1 = v_1e_1v_2e_2 \ldots e_{i-1}v_i$ and $W_2 = v_je_jv_{j+1} \ldots e_{i-1}v_me_mv_0$. Then W_1W_2 is a closed walk in D which has length at least one (since e_m is an edge of W_1W_2) but is shorter than W. This contradicts the choice of W. Hence W is a directed cycle.

(b) Determine whether the analogous statement is true for (undirected) graphs by giving a proof or a counterexample.

Counterexample Let G be the graph with two vertices u, v and one edge $e = uv$. Then $W = ueveu$ is a closed walk in G of length two but G has no cycles. So it is not true for graphs.

Note that the proof in (a) does not work because $W = ueveu$ is a closed walk which repeats an edge but does not repeat a vertex (except when the first vertex is repeated as the last vertex.)

Q4 Let D be a digraph. Prove that

$$\sum_{v\in V(D)} d^-(v) = |A(D)| = \sum_{v\in V(D)} d^+(v).$$

Proof Choose $e \in A(D)$. Since e has a unique tail, e contributes exactly one to $\sum_{v\in V(D)} d^+(v)$. Thus $\sum_{v\in V(D)} d^+(v) = |A(D)|$. The proof that $\sum_{v\in V(D)} d^-(v) = |A(D)|$ is similar.