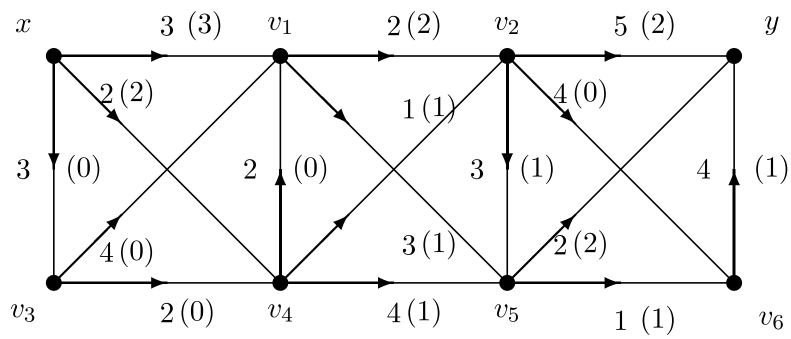
MAS210 Graph Theory Exercises 5

Hand in to BLUE BOX on the GROUND FLOOR of math sci building before 4:30pm on Friday 23/2/07.

Q1 Consider the following directed network N.



The numbers in brackets define an xy-flow f_1 in N. The numbers not in brackets define the capacities of the arcs of N.
(a) Determine the value of f_1.
(b) Grow a maximal f_1-unsaturated tree T_1 rooted at x and use T_1 to construct an xy-flow f_2 in x with $val(f_2) > val(f_1)$.
(c) Grow a maximal f_2-unsaturated tree T_2 rooted at x and use T_2 to construct a set $U \subset V(N)$ with $x \in U$, $y \in V(N) - U$ and $c^+(U) = val(f_2)$. Explain why this equality implies that f_2 is an xy-flow of maximum value in N and $A_N^+(U)$ is an xy-arc cut of minimum capacity. [30]

Q2 Let N be a directed network, $x, y \in V(N)$, and let f be an xy-flow in N. Suppose that P is an f-unsaturated path from x to y in N. Define $g : A(N) \to \mathbb{Z}$ by

$$g(e) = \begin{cases} f(e) & \text{if } e \notin A(P) \\ f(e) + 1 & \text{if } e \text{ is a forward arc of } P \\ f(e) - 1 & \text{if } e \text{ is a backward arc of } P \end{cases}$$

Prove that g is an xy-flow in N and that $val(g) = val(f) + 1$. [15]

Q3 Let N be a directed network, $x, y \in V(N)$, and let f be an xy-flow in N of maximum value. Let U be the set of all vertices which can be reached from x by f-unsaturated paths. Prove that:
(a) $y \notin U$.
(b) All arcs $e \in A_N^+(U)$ satisfy $f(e) = c(e)$ and all arcs $e \in A_N^-(U)$ satisfy $f(e) = 0$.
(c) $val(f) = c^+(U)$. [20]

TURNOVER

Q4 Let D be a digraph and $x, y \in V(D)$. Suppose that $d_D^+(v) = d_D^-(v)$ for all $v \in V(D) - \{x, y\}$. Let $U \subset V(D)$ with $x \in U$ and $y \in V(D) - U$. Let $d_D^+(U)$ be the number of arcs of D from U to $V(D) - U$ and $d_D^-(U)$ be the number of arcs of D from $V(D) - U$ to U.
(a) Prove that $d_D^+(U) - d_D^-(U) = d_D^+(x) - d_D^-(x)$.
(b) Deduce that if $d_D^+(x) > d_D^-(x)$ then there exists a directed path from x to y. *Hint: Let U be the set of all vertices of N which can be reached from x by directed paths and use (a).* [20]

Q5 Let N be an acyclic directed network and $\{x_1, x_2, \ldots, x_n\}$ be an acyclic labeling of D.
(a) Explain why the time that Moravék's algorithm takes to construct the out-arborescence T_i from the out-arborescence T_{i-1} is $O(d_N^-(x_i))$.
(b) Deduce that the total time taken by running Moravék's algorithm to construct a spanning out-arborescence of N is $O(|A(N)|)$. [15]