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B. Sc. Examination By Course Unit

MAS210 Graph Theory and Applications: SPECIMEN EXAM

Duration: 2 hours

Date and time:

This paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.

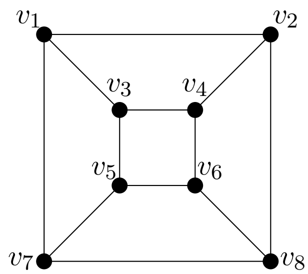
You must not start to read the question paper until instructed to do so by the invigilator.

You must not remove the question paper from the examination room.

 TURN OVER

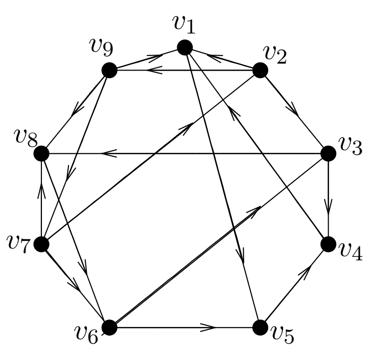
SECTION A *You should attempt all questions. Marks awarded are shown next to the questions.*

Question 1 Consider the following graph G.



An implementation of the breadth first search algorithm produces the tree T_6 with vertices $x_1 = v_1$, $x_2 = v_2$, $x_3 = v_3$, $x_4 = v_7$, $x_5 = v_4$, $x_6 = v_8$, and edges v_1v_2, v_1v_3, v_1v_7, v_2v_4, v_2v_8 at the end of the sixth iteration. List the possible edge(s) which could be added to T_6 in the next iteration. Give a brief description of how the algorithm chooses the edge(s). [5]

Question 2 Consider the following digraph D.



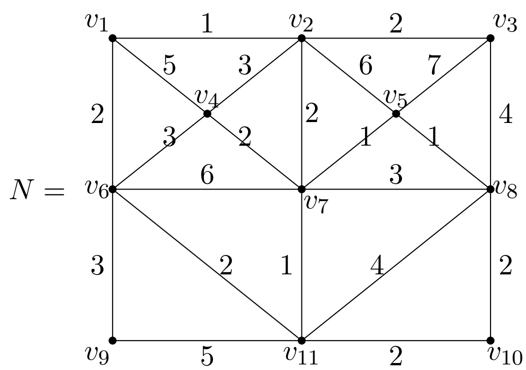
(a) Construct an out-arborescence rooted at v_3 which contains all vertices of D which can be reached from v_3 by directed walks.

(b) Construct an in-arborescence rooted at v_3 which contains all vertices of D which can reach v_3 by directed walks.

(c) Use your solutions to (a) and (b) to determine the strongly connected component of D which contains v_3.

[15]

Question 3 Consider the following network N.

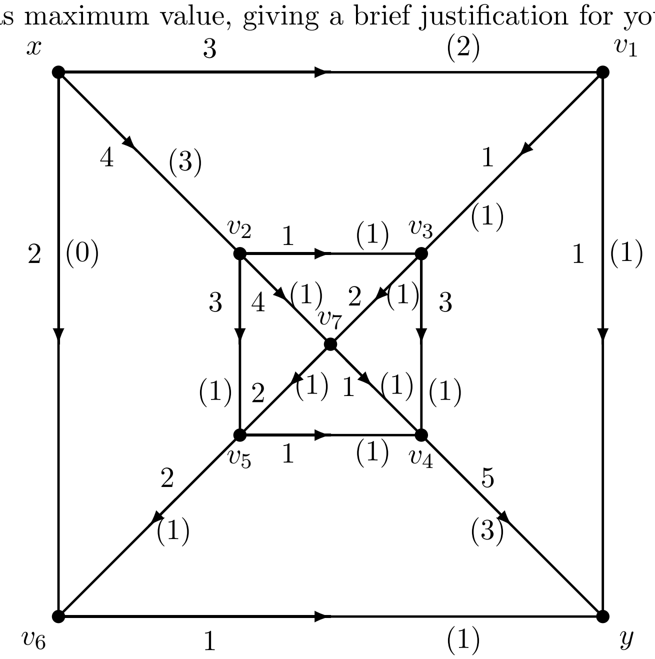


An implementation of Prim's algorithm for finding a minimum weight spanning tree of N produces the following tree T_5 at the end of the fourth iteration: $V(T_5) = \{v_1, v_2, v_3, v_6, v_{11}\}$ and $E(T_5) = \{v_1v_2, v_2v_3, v_1v_6, v_6v_{11}\}$. It also gives the vertex labels shown in the following table.

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}
x_1	x_2	x_3	$[x_2, 3]$	$[x_2, 6]$	x_4	$[x_5, 1]$	$[x_3, 4]$	$[x_4, 3]$	$[x_5, 2]$	x_5

List the edge(s) of N which could be added to T_5 in the next iteration and, for each such edge, give a table showing the new vertex labels. [5]

Question 4 Determine whether the xy-flow given in the following directed network has maximum value, giving a brief justification for your answer.



Numbers in brackets denote the flow $f_1(e)$ along each arc e, numbers not in brackets denote the capacity $c(e)$ of each arc e.

[15]

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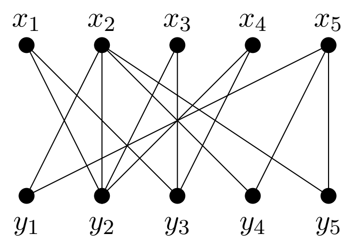
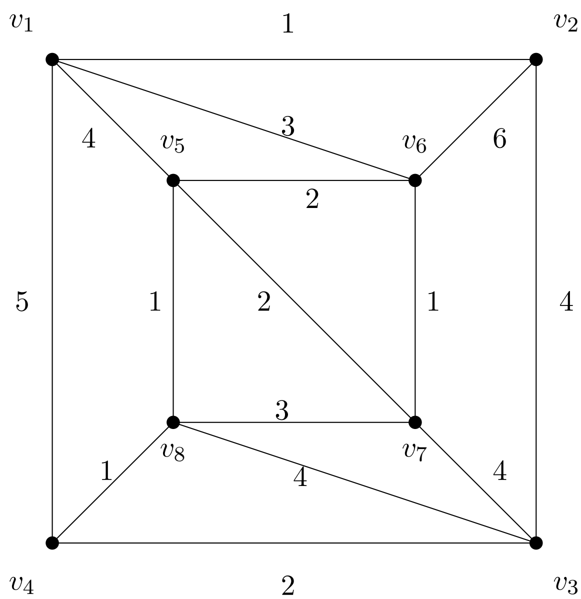
Question 5 Let G be the bipartite graph defined below.



Let $M_1 = \{x_2y_2, x_3y_3, x_5y_5\}$. Use M_1-alternating paths to construct a matching M_2 in G with $|M_2| = 4$. [10]

Question 6 Let N be the network given below.



Let W be a shortest v_4v_8-walk in N which traverses every edge of N at least once. Suppose $w(W) = w(N) + m$. Determine m and give a brief explanation of how you would construct such a walk W. [10]

SECTION B *Each question carries 20 marks. You may attempt all questions. Except for the award of a bare pass, only marks for the best 2 questions will be counted.*

Question 7 (a) Dijkstra's algorithm finds a spanning tree in a network N which contains shortest paths from its root vertex v to all vertices of N by constructing a sequence of trees $T_1, T_2, \ldots, T_n$. Describe how the algorithm constructs the next tree T_{i+1} from the previous tree T_i. (You are not required to go into the details of the vertex labelling procedure.)

(b) Let N be a network and T_i be a tree produced in the i'th iteration of Dijkstra's algorithm applied to N. Prove that the path in T_i from v to x is a shortest path in N from v to x, for all $x \in V(T_i)$.

Question 8 Let N be a directed network with no directed cycles.

(a) Explain what it means to say that a labelling $x_1, x_2, \ldots, x_n$ of the vertices of N is an *acyclic labelling.*

(b) Describe Moravék's algorithm for constructing a spanning out-arborescence rooted at x_1 which contains longest paths from x_1 to all vertices of N.

(c) Assuming that all arithmetic operations take constant time, prove that the time taken in the i'th iteration when Moravék's algorithm is applied to N is $O(d^-_N(x_i))$. Deduce that, under the same assumption, the time taken for Moravék's algorithm to construct a spanning out-arborescence of N is $O(|A(N)|)$.

Question 9 Let N be a network obtained by assigning integer weights to the edges of a complete bipatite graph $K_{n,n}$.

(a) Explain what it means for a map $\ell : V(N) \to \mathbb{Z}$ to be a *feasible vertex labelling.*

(b) Prove that if ℓ is a feasible vertex labelling of N and M is a perfect matching of N then $w(M) \leq \sum_{v \in V(N)} \ell(v)$.

(c) Prove that if M is a maximum weight perfect matching in N, then N has a feasible vertex labelling ℓ such that $w(M) = \sum_{v \in V(N)} \ell(v)$.

 END OF EXAMINATION