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B. Sc. Examination 2007 By Course Unit

MAS210 Graph Theory and Applications: TEST SOLUTIONS

Duration: 45 mins

Date and time: 11am 23 February 2007

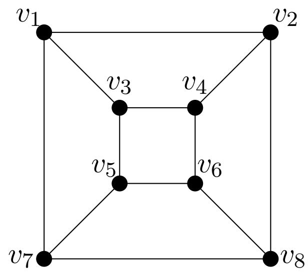
You should attempt all Questions.

You should write your solutions in this booklet, in the space provided after each each question. Additional paper is provided at the end of the booklet for corrected solutions and rough work.

Calculators are NOT permitted in this test. The unauthorised use of a calculator constitutes an examination offence.

 TURN OVER

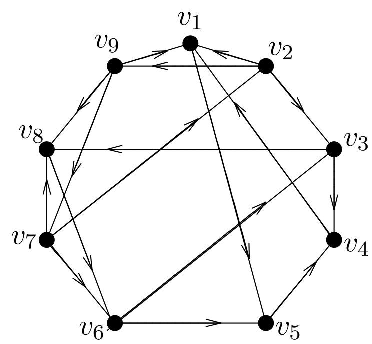
Question 1 *Consider the following graph G.*



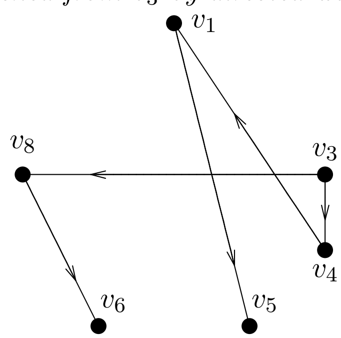
An implementation of the breadth first search algorithm produces the tree T_6 with vertices $x_1 = v_1$, $x_2 = v_2$, $x_3 = v_3$, $x_4 = v_7$, $x_5 = v_4$, $x_6 = v_8$, and edges v_1v_2, v_1v_3, v_1v_7, v_2v_4, v_2v_8 at the end of the sixth iteration. List the possible edge(s) which could be added to T_6 in the next iteration. Give a brief description of how the algorithm chooses the edge(s).

The algorithm chooses the edge v_3v_5.
It chooses an edge from a vertex $x_j \in V(T_6)$ to a vertex $y \in V(G) - V(T_6)$ such that j is as small as possible.

Question 2 *Consider the following digraph D.*



Construct an out-arborescence rooted at v_3 which contains all vertices of D which can be reached from v_3 by directed walks.



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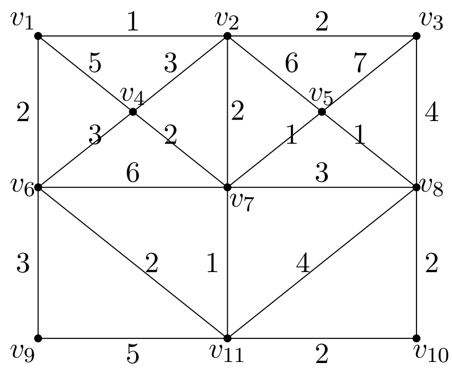
Question 3 *Consider the following network N.*

$N =$



An implementation of Dijkstra's algorithm for finding shortest paths in N starting at v_1 produces the following tree T_5 at the end of the fifth iteration: $V(T_5) = \{v_1, v_2, v_3, v_6, v_7\}$ and $E(T_5) = \{v_1v_2, v_1v_6, v_2v_3, v_2v_7\}$. It also gives the vertex labels shown in the following table.

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}
x_1	x_2	x_4	$[x_2, 4]$	$[x_5, 4]$	x_3	x_5	$[x_5, 6]$	$[x_3, 5]$	$[x_1, \infty]$	$[x_3, 4]$

List the edge(s) of N which could be added to T_5 in the next iteration and, for ONE such edge, give a table showing the new vertex labels.

The algorithm can choose either of the edges $v_2v_4, v_7v_5, v_6v_{11}$. Assuming it chooses v_2v_4, the new vertex labels are

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}
x_1	x_2	x_4	x_6	$[x_5, 4]$	x_3	x_5	$[x_5, 6]$	$[x_3, 5]$	$[x_1, \infty]$	$[x_3, 4]$

Question 4 *Let N be a directed network in which each arc e has been given a non-negative integer capacity $c(e)$, and $x, y \in V(N)$. Explain what it means to say that a map $f : A(N) \to \mathbb{Z}$ is an xy-*flow *in N.*

The map f must satisfy the following two conditions.

- $0 \leq f(e) \leq c(e)$ for all $e \in A(N)$.
- $f^+(v) = f^-(v)$ for all $v \in V(N) - \{x, y\}$, where $f^+(v)$ denotes the sum of the numbers $f(e)$ over all arcs entering v and $f^-(v)$ denotes the sum of the numbers $f(e)$ over all arcs leaving v.

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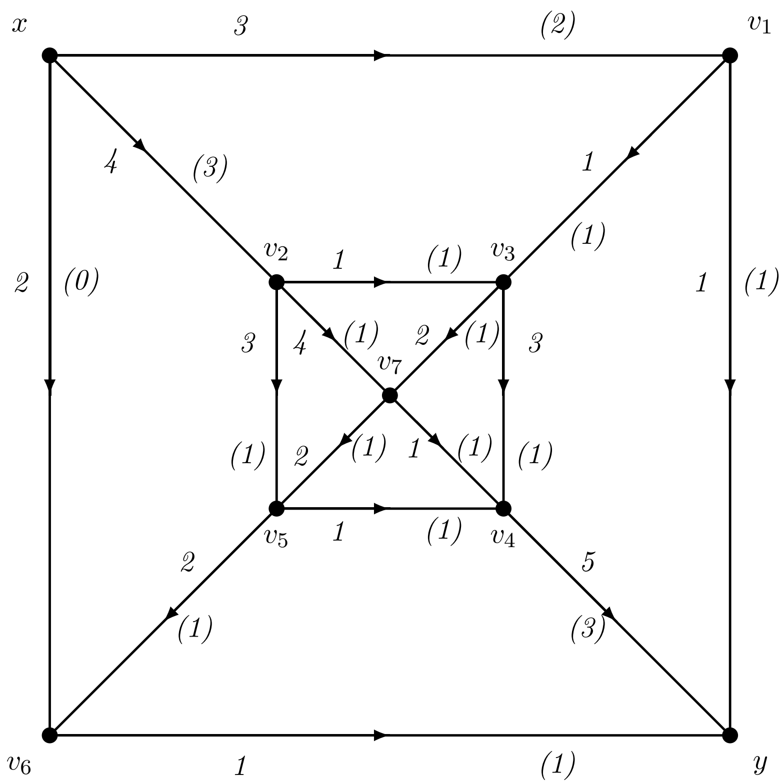
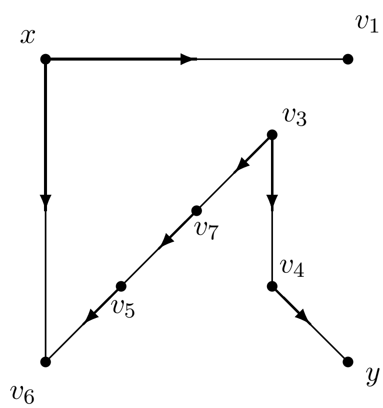
Question 5 *Starting with the given flow f_1, find an xy-flow of maximum value in the following directed network N, giving brief descriptions for the steps in your algorithm. (You may define your maximum flow by updating the numbers in the figure below).*



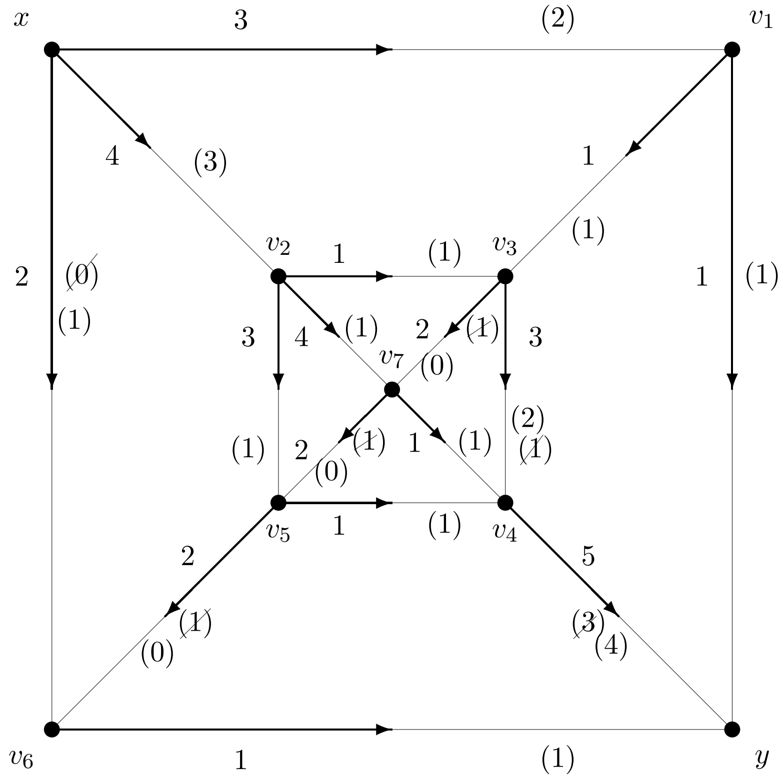
Numbers in brackets denote the flow $f_1(e)$ along each arc e, numbers not in brackets denote the capacity $c(e)$ of each arc e.

Justify the fact that the xy-flow you find has maximum value.

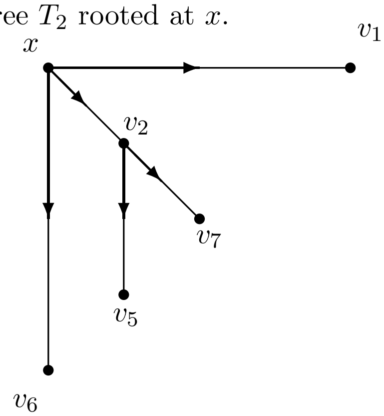
Grow on f_1-unsaturated tree rooted at x.



Send one unit of flow along the f_1-incrementing path $xv_6v_5v_7v_3v_4y$ to create a new flow f_2 shown below.



Grow on f_2-unsaturated tree T_2 rooted at x.



T_2 does not reach y so f_2 is an xy-flow of maximum value.
Putting $U = V(T_2) = \{x, v_1, v_2, v_5, v_6, v_7\}$ we have $val(f_2) = 6 = c^+(U)$. Since for all xy-flows f we have $val(f) \leq c^+(U) = 6$, it follows that f_2 is an xy-flow of maximum value.

 END OF EXAMINATION