TIME SERIES COURSE-WORK 5
SOLUTIONS TO THE THEORY QUESTIONS

Question 5.2.1

(a) $X_t = 0.3X_{t-1} + Z_t$ is an ARMA(p,q) process with $p = 1, q = 0$, i.e, it is AR(1).

In B notation it can be written as $(1 - 0.3B)X_t = Z_t$.

It is a causal process as the root of the associated polynomial $\phi(z) = 1 - 0.3z$ is $z = \frac{10}{3} > 1$, i.e, it is outside the unit interval $[-1, 1]$.

(b) $X_t = Z_t - 1.3Z_{t-1} + 0.4Z_{t-2}$ is an ARMA(p,q) process with $p = 0, q = 2$, i.e., it is MA(2).

In B notation it can be written as $X_t = (1 - 1.3B + 0.4B^2)Z_t$.

To check invertibility we examine roots of the associated polynomial $\theta(z) = 1 - 1.3z + 0.4z^2$. Here we have two real roots $z_1 = 1.25$ and $z_2 = 2$. Both roots are outside the unit interval, hence the process is invertible.

(c) $X_t - 0.5X_{t-1} = Z_t - 1.3Z_{t-1} + 0.4Z_{t-2}$ can be written as $(1-0.5B)X_t = (1-1.3B+0.4B^2)Z_t$. First, however, we need to check if there are common factors. The associated polynomial $\theta(z) = 1 - 1.3z + 0.4z^2$ can be written as

$$(1 - 1.25^{-1}z)(1 - 2^{-1}z) = (1 - 0.8z)(1 - 0.5z).$$

Hence, there is a common factor $1 - 0.5z$ and the model can be simplified. Dividing both sides of the model by $1 - 0.5z$ we obtain

$$X_t = (1 - 0.8B)Z_t.$$

This is an ARMA(p,q) with $p = 0, q = 1$, i.e, it is MA(1). This is an invertible process as the root of the associated polynomial $\theta(z) = 1 - 0.8z$ is outside the unit interval.

Question 5.2.2

(a) $X_t = Z_t + 0.7Z_{t-1}$ represents an invertible MA(1) process with $\theta = 0.7$. For an MA(1) process we have the following formulae for the ACF and the PACF, respectively.

$$\rho(\tau) = \begin{cases} \frac{\theta}{1+\theta^2} & \text{for } \tau = 1 \\ 0 & \text{for } \tau > 1 \end{cases}$$

$$\phi_{\tau\tau} = -\frac{(-\theta)^\tau(1 - \theta^2)}{1 - \theta^{2(\tau+1)}}, \quad \text{for } \tau \geq 1.$$

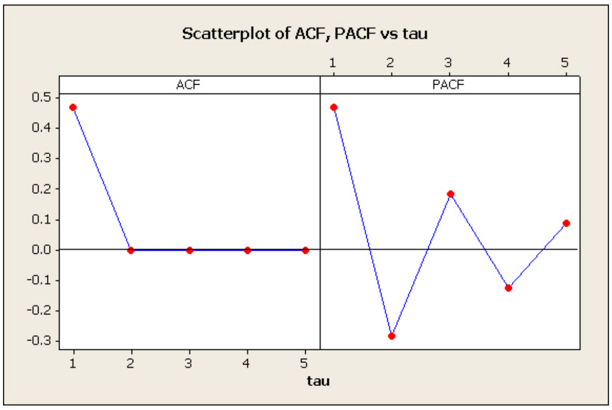
For $\theta = 0.7$ we obtain

$$\rho(\tau) = \begin{cases} 0.47 & \text{for } \tau = 1 \\ 0 & \text{for } \tau > 1 \end{cases}$$

and

$$\phi_{11} = 0.47, \phi_{22} = -0.28, \phi_{33} = 0.19, \phi_{44} = -0.13, \phi_{55} = 0.09.$$

The ACF cuts off after lag 1, PACF alternates sign and tails off.



(b) $X_t = 0.9X_{t-1} - 0.2X_{t-2} + Z_t$ represents a causal AR(2) process with $\phi_1 = 0.9$ and $\phi_2 = -0.2$. To obtain the ACF we may use the difference equations of order two (as in Example 6.4 of Lecture Notes). Then the ACF is

$$\rho(\tau) = c_1 z_1^{-\tau} + c_2 z_2^{-\tau},$$

where the roots of the associated polynomial $\phi(z) = 1 - 0.9z + 0.2z^2$ are $z_1 = 2$ and $z_2 = 2.5$ and the constants c_1, c_2 can be found from the initial conditions and are equal to $c_1 = 3.5$ and $c_2 = -2.5$. Hence, we obtain

$$\rho(\tau) = 3.5 \times 2^{-\tau} - 2.5 \times 2.5^{-\tau}, \quad \text{for } \tau = 1, 2, \ldots$$

The first five values of the ACF are

$$\rho(1) = 0.75, \rho(2) = 0.48, \rho(3) = 0.28, \rho(4) = 0.15, \rho(5) = 0.08.$$

From Remark 6.11 of the Lecture Notes we have the PACF for AR(2) equal to

$$\begin{aligned}
\phi_{11} =& \frac{\phi_1}{1 - \phi_2} = 0.75 \\
\phi_{22} =& \phi_2 = -0.2 \\
\phi_{\tau\tau} =& 0 \ \text{ for } \tau > 2
\end{aligned}$$

For AR(2) the ACF tails off while the PACF cuts off after lag 2.