TIME SERIES COURSE-WORK 3
SOLUTIONS TO THE THEORY QUESTIONS

Question 1
An MA(2) model is

$$X_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2}, \quad \text{where, } Z_t \sim WN(0, \sigma^2).$$

To derive the ACVF and then the ACF of the MA(2) we calculate $\text{cov}(X_t, X_{t+\tau})$ for all $t, \tau = 0, \pm 1, \pm 2, \ldots$.

$$\begin{aligned}\text{cov}(X_t, X_{t+\tau}) &= \text{cov}(Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2}, Z_{t+\tau} + \theta_1 Z_{t+\tau-1} + \theta_2 Z_{t+\tau-2}) \\ &= \text{E}[(Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2})(Z_{t+\tau} + \theta_1 Z_{t+\tau-1} + \theta_2 Z_{t+\tau-2})]\end{aligned}$$

as $\text{E}(Z_t) = 0$ for all t. This gives

$$\begin{aligned}\text{cov}(X_t, X_{t+\tau}) = \text{E}[&Z_t Z_{t+\tau} + \theta_1 Z_t Z_{t+\tau-1} + \theta_2 Z_t Z_{t+\tau-2} \\ &+ \theta_1 Z_{t-1} Z_{t+\tau} + \theta_1^2 Z_{t-1} Z_{t+\tau-1} + \theta_1\theta_2 Z_{t-1} Z_{t+\tau-2} \\ &+ \theta_2 Z_{t-2} Z_{t+\tau} + \theta_1\theta_2 Z_{t-2} Z_{t+\tau-1} + \theta_2^2 Z_{t-2} Z_{t+\tau-2}]\end{aligned}$$

Hence, as Z_t are uncorrelated random variables with $\text{var}(Z_t) = \sigma^2$, we obtain

$$\gamma(\tau) = \begin{cases} (1 + \theta_1^2 + \theta_2^2)\sigma^2 & \text{for} \quad \tau = 0 \\ (\theta_1 + \theta_1\theta_2)\sigma^2 & \text{for} \quad \tau = \pm 1 \\ \theta_2\sigma^2 & \text{for} \quad \tau \pm 2 \\ 0 & \text{for} \quad |\tau| > 2 \end{cases}$$

Dividing each of these expressions by $\gamma(0)$ we obtain the autocorrelation function $\rho(\tau)$

$$\rho(\tau) = \begin{cases} 1 & \text{for} \quad \tau = 0 \\ \frac{(\theta_1 + \theta_1\theta_2)}{1 + \theta_1^2 + \theta_2^2} & \text{for} \quad \tau = \pm 1 \\ \frac{\theta_2}{1 + \theta_1^2 + \theta_2^2} & \text{for} \quad \tau \pm 2 \\ 0 & \text{for} \quad |\tau| > 2 \end{cases}$$

Question 2
First note that $\text{E}(Z_t) = 0$ for all t and so $\text{E}(X_t) = 0$ for all t. Then the autocovariance is

$$\text{cov}(X_t, X_{t+\tau}) = \frac{1}{(q+1)^2} \text{E}[\sum_{k=0}^{q} Z_{t-k} \sum_{j=0}^{q} Z_{t+\tau-j}].$$

For $\tau = 0$ we obtain

$$\gamma(0) = \frac{1}{(q+1)^2}(q+1)\sigma^2 = \frac{\sigma^2}{q+1}$$

as Z_t are uncorrelated and there are exactly $q + 1$ products $Z_j Z_j$ (with same indexes) and their expectation is $\text{E}(Z_j^2) = \sigma^2$. Similarly, there will be $q + 1 - \tau$ pairs $Z_j Z_j$ for $\tau = 1, 2, \ldots, q$ and none for $\tau > q$. Hence

$$\gamma(\tau) = \frac{1}{(q+1)^2}(q + 1 - \tau)\sigma^2 \text{ for } \tau = 1, 2, \ldots, q.$$

Hence

$$\rho(\tau) = \begin{cases} \frac{(q+1-\tau)\sigma^2}{(q+1)^2} \frac{q+1}{\sigma^2} = \frac{q+1-\tau}{q+1} & \text{for} \quad \tau = 0, 1, 2, \ldots, q \\ 0 & \text{for} \quad \tau > q. \end{cases}$$

□

Question 3

Additive model $X_t = m_t + s_t + Y_t$

We assume that $s_t = s_{t-12}$ and $m_t = \beta_0 + \beta_1 t$. Then

$$\begin{aligned}\nabla_{12}X_t =& X_t - X_{t-12}\\ =& m_t + s_t + Y_t - (m_{t-12} + s_{t-12} + Y_{t-12})\\ =& m_t - m_{t-12} + \nabla_{12}Y_t\\ =& \beta_0 + \beta_1 t - (\beta_0 + \beta_1(t-12)) + \nabla_{12}Y_t\\ =& 12\beta_1 + \nabla_{12}Y_t.\end{aligned}$$

To check stationarity we check if the expectation and variance are constant and the covariances do not depend on t.

$\mathrm{E}(\nabla_{12}X_t) = 12\beta_1$, constant.

$$\begin{aligned}\mathrm{cov}(\nabla_{12}X_t, \nabla_{12}X_{t+\tau}) =& \mathrm{cov}(\nabla_{12}Y_t, \nabla_{12}Y_{t+\tau})\\ =& \mathrm{cov}(Y_t - Y_{t-12}, Y_{t+\tau} - Y_{t+\tau-12})\\ =& \mathrm{cov}(Y_t, Y_{t+\tau}) - \mathrm{cov}(Y_{t-12}, Y_{t+\tau}) - \mathrm{cov}(Y_t, Y_{t+\tau-12}) + \mathrm{cov}(Y_{t-12}, Y_{t+\tau-12})\\ =& 2\gamma_Y(\tau) - \gamma_Y(\tau+12) - \gamma_Y(\tau-12).\end{aligned}$$

This does not depend on t, hence the differenced series is stationary.

Multiplicative model $X_t = m_t s_t + Y_t$

Here we have

$$\begin{aligned}\nabla_{12}X_t =& m_t s_t + Y_t - (m_{t-12}s_{t-12} + Y_{t-12})\\ =& (\beta_0 + \beta_1 t)s_t - (\beta_0 + \beta_1(t-12))s_{t-12} + \nabla_{12}Y_t\\ =& 12\beta_1 s_{t-12} + \nabla_{12}Y_t\end{aligned}$$

It gives

$$\mathrm{E}(\nabla_{12}X_t) = 12\beta_1 s_{t-12}.$$

This still depends on t, it is not a constant. However, as $s_t = s_{t-12} = s_{t-24}$, we should eliminate the seasonality effect from $\nabla_{12}X_t$ by applying the same operator again, that is by $\nabla_{12}(\nabla_{12}X_t)$, as follows

$$\begin{aligned}\nabla_{12}^2 = \nabla_{12}(\nabla_{12}X_t) =& 12\beta_1 s_{t-12} + \nabla_{12}Y_t - (12\beta_1 s_{t-24} + \nabla_{12}Y_{t-12})\\ =& Y_t - 2Y_{t-12} + Y_{t-24}.\end{aligned}$$

Then $\mathrm{E}(\nabla_{12}^2 X_t) = 0$ and $\mathrm{cov}(\nabla_{12}^2 X_t), \nabla_{12}^2 X_{t+\tau})$ is a constant, what can be easily shown, similarly as in the previous part. Hence $\nabla_{12}(\nabla_{12}X_t) = \nabla_{12}^2 X_t$ is a stationary process.

Question 4

See lecture notes, Chapter 4.6.