TIME SERIES COURSE-WORK 2
SOLUTIONS TO THE THEORY QUESTIONS

Question 1
The definition of covariance and simple properties of expectation give:

$$\begin{aligned}\operatorname{cov}(X_1, X_2) &= \mathrm{E}[(X_1 - \mathrm{E}(X_1))(X_2 - \mathrm{E}(X_2))] \\ &= \mathrm{E}[X_1X_2 - X_1\,\mathrm{E}(X_2) - X_2\,\mathrm{E}(X_1) + \mathrm{E}(X_1)\,\mathrm{E}(X_2)] \\ &= \mathrm{E}(X_1X_2) - \mathrm{E}(X_1)\,\mathrm{E}(X_2) - \mathrm{E}(X_2)\,\mathrm{E}(X_1) + \mathrm{E}(X_1)\,\mathrm{E}(X_2) \\ &= \mathrm{E}(X_1X_2) - \mathrm{E}(X_1)\,\mathrm{E}(X_2)\end{aligned}$$

□

Question 2
There are various ways to show that $-1 \leq \rho(X_1, X_2) \leq 1$. Here we use the fact that (by definition) variance of any random variable cannot be negative.

For a combination of two random variables we may write:

$$0 \leq \operatorname{var}(\lambda X_1 + X_2) = \lambda^2 \operatorname{var}(X_1) + 2\lambda \operatorname{cov}(X_1X_2) + \operatorname{var}(X_2)$$

for any real constant λ. This may be treated as a quadratic function in λ, which must be nonnegative. Hence, the discriminant $\Delta \leq 0$, that is

$$\Delta = 4[\operatorname{cov}(X_1X_2)]^2 - 4\operatorname{var}(X_1)\operatorname{var}(X_2) \leq 0.$$

This gives

$$[\operatorname{cov}(X_1X_2)]^2 \leq \operatorname{var}(X_1)\operatorname{var}(X_2)$$

or

$$|\operatorname{cov}(X_1X_2)| \leq \sqrt{\operatorname{var}(X_1)}\sqrt{\operatorname{var}(X_2)}.$$

Hence

$$|\rho(X_1X_2)| = |\frac{\operatorname{cov}(X_1X_2)}{\sqrt{\operatorname{var}(X_1)}\sqrt{\operatorname{var}(X_2)}}| \leq 1.$$

□

Question 3

(a) The marginal distributions of X_1 and of X_2 are, respectively,

x_1	51	52	53	54	55
$P(X_1 = x_1)$	0.28	0.28	0.22	0.09	0.13

x_2	51	52	53	54	55
$P(X_2 = x_2)$	0.18	0.15	0.35	0.12	0.20

(b) Use formulae (3.23) and (3.27) of Chapter 3 (Lecture Notes). Here we have $P(X_1 = 55) = 0.13$ and we obtain

$$\mathrm{E}(X_2|X_1 = 55) = \frac{1}{0.13}(51\times0.01+52\times0.01+53\times0.05+54\times0.03+55\times0.03) = 53.46.$$

Expected sale of aspirin in September by the neighborhood drugstore is 53.46.

Question 4

The joint density function of random variables X_1 and X_2 is given by

$$f(x_1, x_2) = \begin{cases} 4x_1x_2e^{-(x_1^2+x_2^2)}, & \text{for } 0 \le x_1 < \infty, 0 \le x_2 < \infty \\ 0, & \text{otherwise} \end{cases}$$

(a) Marginal density function of X_1 is

$$f_{X_1}(x_1) = \int_{-\infty}^{\infty} f(x_1, x_2)dx_2 = \int_0^{\infty} 4x_1x_2e^{-(x_1^2+x_2^2)}dx_2.$$

Knowing that the derivative of $e^{-(x_1^2+x_2^2)}$ with respect to x_2 is

$$\frac{\partial e^{-(x_1^2+x_2^2)}}{\partial x_2} = -2x_2e^{-(x_1^2+x_2^2)}$$

we can write

$$\begin{aligned}\int_0^{\infty} 4x_1x_2e^{-(x_1^2+x_2^2)}dx_2 =& -2x_1\int_0^{\infty} -2x_2e^{-(x_1^2+x_2^2)}dx_2 \\ =& -2x_1\left(e^{-(x_1^2+x_2^2)}\right)_0^{\infty} = -2x_1(-e^{-x_1^2}) \\ =& 2x_1e^{-x_1^2}.\end{aligned}$$

Hence,

$$f_{X_1}(x_1) = 2x_1e^{-x_1^2}$$

Similarly, we obtain

$$f_{X_2}(x_2) = 2x_2e^{-x_2^2}$$

(b) Conditional density function of X_1 given $X_2 = x_2$ is

$$f_{X_1}(x_1|x_2) = \frac{f_{X_1,X_2}(x_1, x_2)}{f_{X_2}(x_2)} = \frac{4x_1x_2e^{-(x_1^2+x_2^2)}}{2x_2e^{-x_2^2}} = 2x_1e^{-x_1^2}$$

Similarly, we obtain

$$f_{X_2}(x_2|x_1) = 2x_2e^{-x_2^2}$$

(c)The two variables are independent and so

$$\begin{aligned}\mathrm{E}(X_1|X_2 = x_2) =& \mathrm{E}(X_1) \\ =& \int_0^{\infty} x_1^2e^{-x_1^2}dx_1 \\ =& \left(-x_1e^{-x_1^2}\right)_0^{\infty} + \int_0^{\infty} e^{-x_1^2}dx_1 \qquad \text{(by parts)} \\ =& 0 + \int_0^{\infty} e^{-x_1^2}dx_1 \\ =& \frac{1}{\sqrt{2}}\int_0^{\infty} e^{-\frac{z^2}{2}}dz \qquad \left(\text{by substitution } x = \frac{z}{\sqrt{2}}\right) \\ =& \sqrt{\pi}\frac{1}{\sqrt{2\pi}}\int_0^{\infty} e^{-\frac{z^2}{2}}dz \qquad \text{(standard normal distribution)} \\ =& \frac{\sqrt{\pi}}{2}\end{aligned}$$

Similarly for X_2.