6.2 ACF and PACF of ARMA(p,q)

6.2.1 ACF of ARMA(p,q)

In Section 4.6 we have derived the ACF for ARMA(1,1) process. We have used the linear process representation and the fact that

$$\gamma(\tau) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+\tau}.$$

We have calculated the coefficients ψ_j from the relation

$$\psi(B) = \frac{\theta(B)}{\phi(B)},$$

which (in case of ARMA(1,1)) gives the values

$$\psi_j = \phi_1^{j-1}(\theta_1 + \phi_1).$$

This allows us to calculate the ACF of the process

$$\rho(\tau) = \frac{\gamma(\tau)}{\gamma(0)}.$$

Another way of finding the coefficients ψ is using the homogeneous difference equations. However, we may obtain such equation directly in terms of $\gamma(\tau)$ or $\rho(\tau)$.

For ARMA(1,1)

$$X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1}$$

we can write

$$\begin{aligned}
\gamma(\tau) &= \operatorname{cov}(X_{t+\tau}, X_t) \\
&= \mathrm{E}(X_{t+\tau}X_t) \\
&= \mathrm{E}[(\phi X_{t+\tau-1} + Z_{t+\tau} + \theta Z_{t+\tau-1})X_t] \\
&= \mathrm{E}[\phi X_{t+\tau-1}X_t + Z_{t+\tau}X_t + \theta Z_{t+\tau-1}X_t] \\
&= \phi\, \mathrm{E}[X_{t+\tau-1}X_t] + \mathrm{E}[Z_{t+\tau}X_t] + \theta\, \mathrm{E}[Z_{t+\tau-1}X_t]
\end{aligned}$$

Here we consider a causal ARMA(1,1) process, hence

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}.$$

This gives

$$\begin{aligned}
\mathrm{E}[Z_{t+\tau}X_t] &= \mathrm{E}[Z_{t+\tau}\sum_{j=0}^{\infty}\psi_j Z_{t-j}] \\
&= \sum_{j=0}^{\infty}\psi_j \,\mathrm{E}[Z_{t+\tau}Z_{t-j}] \\
&= \begin{cases} \psi_0\sigma^2 & \text{for } \tau = 0, \\ 0 & \text{for } \tau \geq 1. \end{cases}
\end{aligned}$$

Also,

$$\begin{aligned}
\mathrm{E}[Z_{t+\tau-1}X_t] &= \mathrm{E}[Z_{t+\tau-1}\sum_{j=0}^{\infty}\psi_j Z_{t-j}] \\
&= \sum_{j=0}^{\infty}\psi_j \,\mathrm{E}[Z_{t+\tau-1}Z_{t-j}] \\
&= \begin{cases} \psi_1\sigma^2 & \text{for } \tau = 0, \\ \psi_0\sigma^2 & \text{for } \tau = 1 \\ 0 & \text{for } \tau \geq 2. \end{cases}
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\psi_0 &= 1 \\
\psi_1 &= \phi + \theta.
\end{aligned}$$

Putting all these together we obtain

$$\begin{aligned}
\gamma(\tau) &= \phi\,\mathrm{E}[X_{t+\tau-1}X_t] + \mathrm{E}[Z_{t+\tau}X_t] + \theta\,\mathrm{E}[Z_{t+\tau-1}X_t] \\
&= \begin{cases} \phi\gamma(1) + \sigma^2(1 + \phi\theta + \theta^2) & \text{for } \tau = 0, \\ \phi\gamma(0) + \sigma^2\theta & \text{for } \tau = 1, \\ \phi\gamma(\tau - 1) & \text{for } \tau \geq 2. \end{cases}
\end{aligned}$$

The ACVF is in fact given here in the form of a homogeneous difference equation of order 1 with initial conditions specifying $\gamma(0)$ and $\gamma(1)$. Namely, we have

$$\gamma(\tau) - \phi\gamma(\tau - 1) = 0 \tag{6.11}$$

and the initial conditions are

$$\begin{cases} \gamma(0) = \phi\gamma(1) + \sigma^2(1 + \phi\theta + \theta^2) \\ \gamma(1) = \phi\gamma(0) + \sigma^2\theta \end{cases} \tag{6.12}$$

Note that the equation (6.11)

$$\gamma(\tau) = \phi\gamma(\tau - 1)$$

has an iterative form and we can write

$$\begin{aligned}
\gamma(2) &= \phi\gamma(1)\\
\gamma(3) &= \phi\gamma(2) = \phi^2\gamma(1)\\
\gamma(4) &= \phi\gamma(3) = \phi^3\gamma(1)\\
&\dots\\
\gamma(\tau) &= \phi^{\tau-1}\gamma(1)
\end{aligned}$$

The polynomial associated with the equation (6.11) is

$$1 - \phi z = 0$$

with root

$$z_0 = \frac{1}{\phi}.$$

So we can write

$$\gamma(\tau) = (z_0^{-1})^{\tau-1}\gamma(1).$$

This depends only on the root of the associated polynomial and on the initial conditions. Solving (6.12) for $\gamma(0)$ and $\gamma(1)$ we obtain

$$\gamma(0) = \sigma^2\frac{1+2\theta\phi+\theta^2}{1-\phi^2}$$

and

$$\gamma(1) = \sigma^2\frac{(1+\theta\phi)(\phi+\theta)}{1-\phi^2}$$

This gives us

$$\gamma(\tau) = \sigma^2\frac{(1+\theta\phi)(\phi+\theta)}{1-\phi^2}\phi^{\tau-1}, \quad \text{for } \tau \geq 1.$$

Finally dividing by $\gamma(0)$ we get the ACF, which is the same as the one derived in Section 4.6, that is

$$\rho(\tau) = \frac{(1+\theta\phi)(\phi+\theta)}{1+2\theta\phi+\theta^2}\phi^{\tau-1}, \quad \text{for } \tau \geq 1. \tag{6.13}$$

ACF for ARMA(p,q)

Assume that the model

$$\phi(B)X_t = \theta(B)Z_t$$

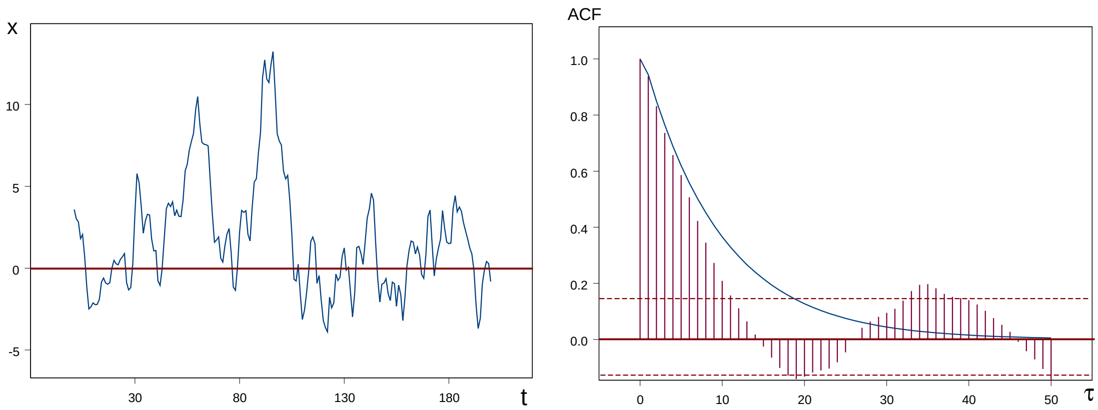

Figure 6.1: ARMA(1,1) simulated process $x_t - 0.9x_{t-1} = z_t + 0.5z_{t-1}$, sample ACF and the theoretical ACF of this process.

is causal, that is the roots of $\phi(B)$ are outside the unit circle. Then we can write

$$X_t = \psi(B)Z_t,$$

where

$$\psi(B) = \sum_{j=0}^{\infty} \psi_j B^j$$

and it follows immediately that $\mathrm{E}(X_t) = 0$.

As in the example for ARMA(1,1), we can obtain a homogeneous differential equation in terms of $\gamma(\tau)$ with some initial conditions. Namely

$$\begin{aligned}
\gamma(\tau) &= \operatorname{cov}(X_{t+\tau}, X_t) \\
&= \mathrm{E}\left[\left(\sum_{j=1}^{p} \phi_j X_{t+\tau-j} + \sum_{j=0}^{q} \theta_j Z_{t+\tau-j}\right) X_t\right] \\
&= \sum_{j=1}^{p} \phi_j \,\mathrm{E}[X_{t+\tau-j}X_t] + \sum_{j=0}^{q} \theta_j \,\mathrm{E}[Z_{t+\tau-j}X_t] \\
&= \sum_{j=1}^{p} \phi_j \gamma(\tau - j) + \sigma^2 \sum_{j=\tau}^{q} \theta_j \psi_{j-\tau}
\end{aligned}$$

Here, as before, we used the linear representation of X_t, the fact that Z_{t+i} and X_t are uncorrelated for $i > 0$, $\psi_i = 0$ for $i < 0$ and that $\theta_0 = 1$.

This gives the general homogeneous difference equation for $\gamma(\tau)$,

$$\gamma(\tau) - \phi_1\gamma(\tau-1) - \ldots - \phi_p\gamma(\tau - p) = 0 \quad \text{for } \tau \geq \max(p, q+1), \tag{6.14}$$

with initial conditions

$$\gamma(\tau)-\phi_1\gamma(\tau-1)-\ldots-\phi_p\gamma(\tau-p)=\sigma^2(\theta_\tau\psi_0+\theta_{\tau+1}\psi_1+\ldots+\theta_q\psi_{q-\tau}) \quad (6.15)$$

for $0 \leq \tau < \max(p, q+1)$.

Example 6.4. **ACF of an AR(2) process**
Let

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} = Z_t$$

be a causal AR(2) process. From (6.14) we have

$$\gamma(\tau) - \phi_1\gamma(\tau-1) - \phi_2\gamma(\tau-2) = 0 \quad \text{for } \tau \geq 2$$

with initial conditions

$$\begin{cases} \gamma(0) - \phi_1\gamma(-1) - \phi_2\gamma(-2) = \sigma^2 \\ \gamma(1) - \phi_1\gamma(0) - \phi_2\gamma(-1) = 0 \end{cases}$$

It is convenient to write these equations in terms of the autocorrelation function $\rho(\tau)$. Dividing them by $\gamma(0)$ we obtain

$$\begin{cases} \rho(\tau) - \phi_1\rho(\tau-1) - \phi_2\rho(\tau-2) = 0, \quad \text{for } \tau \geq 2 \\ \rho(0) = 1 \\ \rho(1) = \dfrac{\phi_1}{1-\phi_2} \end{cases} \quad (6.16)$$

We know that a general solution to a second order difference equation is

$$\rho(\tau) = c_1 z_1^{-\tau} + c_2 z_2^{-\tau}$$

where z_1 and z_2 are the roots of the associated polynomial

$$\phi(z) = 1 - \phi_1 z - \phi_2 z^2,$$

and c_1 and c_2 can be found from the initial conditions.

Take $\phi_1 = 0.7$ and $\phi_2 = -0.1$, that is the AR(2) process is

$$X_t - 0.7X_{t-1} + 0.1X_{t-2} = Z_t.$$

It is a causal process as the coefficients lie in the admissible parameter space. Also, the roots of the associated polynomial

$$\phi(z) = 1 - 0.7z + 0.1z^2$$

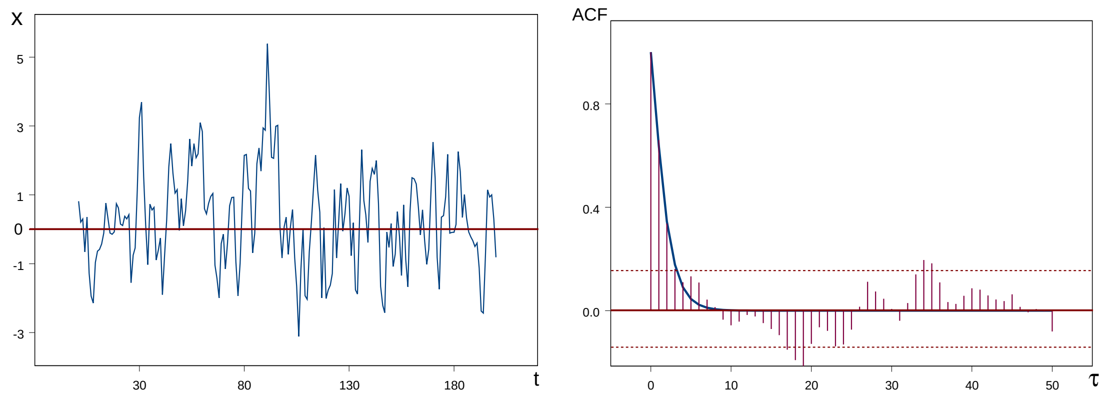

Figure 6.2: AR(2) simulated process $x_t - 0.7x_{t-1} + 0.1x_{t-2} = z_t$, sample ACF and the theoretical ACF of this process.

are $z_1 = 2$ and $z_2 = 5$, i.e., they are outside the unit circle. The initial conditions are

$$\begin{cases} \rho(0) = 1 \\ \rho(1) = \dfrac{0.7}{1+0.1} = \dfrac{7}{11} \end{cases}$$

They give the set of equations for c_1 and c_2, namely

$$\begin{cases} c_1 + c_2 = 1 \\ \dfrac{1}{2}c_1 + \dfrac{1}{5}c_2 = \dfrac{7}{11} \end{cases}$$

These give

$$c_1 = \frac{16}{11}, \quad c_2 = -\frac{5}{11}$$

and finally we obtain the ACF for this AR(2) process

$$\rho(\tau) = \frac{16}{11}2^{-\tau} - \frac{5}{11}5^{-\tau} = \frac{2^{4-\tau} - 5^{1-\tau}}{11}.$$

Simulated AR(2) process, its sample ACF and the theoretical ACF are shown in Figure 6.2. As we can see, the theoretical ACF decreases quickly towards zero, but it never attains zero, we say it tails off.