Chapter 6

ARMA Models

6.1 ARMA Processes

In Section (4.6) we have introduced a special case (for $p = 1$ and $q = 1$) of a very general class of stationary TS models called **Autoregressive Moving Average (ARMA) Models**. In this section we will consider this class of models for general values of the model orders p and q.

Definition 6.1. $\{X_t\}$ *is an **ARMA(p,q)** process if* $\{X_t\}$ *is stationary and if for every* t,

$$X_t - \phi_1 X_{t-1} - \ldots - \phi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + \ldots + \theta_q Z_{t-q}, \qquad (6.1)$$

where $\{Z_t\} \sim WN(0, \sigma^2)$ *and the polynomials*

$$\phi(z) = 1 - \phi_1 z - \ldots - \phi_p z^p \qquad (6.2)$$

and

$$\theta(z) = 1 + \theta_1 z + \ldots + \theta_q z^q \qquad (6.3)$$

have no common factors.

Remark 6.1. The process $\{X_t\}$ is said to be an **ARMA(p,q) process with mean** μ if $\{X_t - \mu\}$ is an ARMA(p,q) process.

Remark 6.2. If the polynomials (6.2) and (6.3) have no common factors, it means that the model can not be reduced to a simpler one. If the polynomials do have common factors then there are redundant parameters what unnecessarily complicates further analysis of the model. Then the model should be simplified.

Using the backshift operator we can write the equation (6.1) concisely as

$$\phi(B)X_t = \theta(B)Z_t, \qquad (6.4)$$

where $\phi(B)$ and $\theta(B)$ are the regressive operator (polynomial in B) and the moving average operator (polynomial in B) of the form (6.2) and (6.3), respectively. Note that when $\phi(B) = 1$ then ARMA(p,q) is equivalent to MA(q) and when $\theta(B) = 1$ then ARMA(p,q) is equivalent to AR(p). Such processes are often denoted as ARMA(0,q) and ARMA(p,0) to stress the fact that the moving average model and the autoregressive model are members of the ARMA models family.

In Section (4.6) we discussed causality and invertibility of ARMA(1,1). These two properties are related to the solution X_t of (6.4) being represented as a combination of past noise values and the solution Z_t of (6.4) being represented as a combination of past X_t variables, respectively.

6.1.1   Causality of ARMA(p,q)

We showed that the condition for stationarity of ARMA(1,1)

$$X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1}, \quad \text{for every } t,$$

is that

$$|\phi| \neq 1,$$

that is $1 - \phi \neq 0$ or $1 + \phi \neq 0$. This is equivalent to say that the polynomial

$$\phi(z) = 1 - \phi z \neq 0 \ \text{ for } |z| = 1.$$

We have also derived the condition for causality of ARMA(1,1), which is

$$|\phi| < 1.$$

This condition can be viewed in terms of the solution to the equation

$$\phi(z) = 1 - \phi z = 0$$

which is $z = \frac{1}{\phi}$ and which should be bigger than 1 or smaller than -1.

Similar conditions, which are given in the following proposition, are put on ARMA(p,q).

Proposition 6.1. *A stationary solution* $\{X_t\}$ *of equation (6.4) exists if and only if*

$$\phi(z) = 1 - \phi_1 z - \ldots - \phi_p z^p \neq 0 \ \ \text{for all } |z| = 1.$$

The process is causal, that is there exist constants $\{\psi_j\}$ *such that* $\sum_{j=0}^{\infty} |\psi_j| < \infty$ *and*

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}, \tag{6.5}$$

if and only if

$$\phi(z) = 1 - \phi_1 z - \ldots - \phi_p z^p = 0 \quad \text{only for } |z| > 1.$$

Example 6.1. **Causality of AR(2)**

For AR(1) process it is easy to establish the relation between the causality condition, $|\phi| < 1$, and the roots of the polynomial $1 - \phi z$ which are $\frac{1}{\phi}$. It is not that easy to see the relation between the two, that is between the values of the parameters $\phi_1, \ldots, \phi_p$ and the zeros of the polynomial $1 - \phi_1 z - \ldots - \phi_p z^p$ for large p.

For AR(2), which can be written as

$$(1 - \phi_1 B - \phi_2 B^2) X_t = Z_t,$$

to be causal we require that the roots of the polynomial

$$\phi(z) = 1 - \phi_1 z - \phi_2 z^2$$

lie outside the unit circle $|z| = 1$. This requirement can be written as

$$\left| \frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{-2\phi_2} \right| > 1.$$

Remark 6.3. Note that the causality conditions for AR(p) and ARMA(p,q) are the same.

The sequence $\{\psi_j\}$ in (6.5) can be derived from the relation (6.4), that is

$$X_t = \frac{\theta(B)}{\phi(B)} Z_t = \psi(B) Z_t.$$

That is

$$\phi(B)\psi(B) = \theta(B). \tag{6.6}$$

In terms of polynomials in z we may write the identity

$$(1 - \phi_1 z - \ldots - \phi_p z^p)(\psi_0 + \psi_1 z + \ldots) = 1 + \theta_1 z + \ldots + \theta_q z^q.$$

Equating the coefficients of z^j, $j = 0, 1, \ldots$, we obtain

$$1 = \psi_0$$
$$\theta_1 = \psi_1 - \psi_0 \phi_1$$
$$\theta_2 = \psi_2 - \psi_1 \phi_1 - \psi_0 \phi_2$$
$$\ldots$$
$$\theta_j = \psi_j - \sum_{k=1}^{p} \phi_k \psi_{j-k}.$$

In the last expression we have $\theta_0 = 1$, $\theta_j = 0$ for $j > q$, and $\psi_j = 0$ for $j < 0$.

Example 6.2. Consider ARMA(2,1)

$$X_t - 0.8X_{t-1} - 0.1X_{t-2} = Z_t + 0.3Z_{t-1}.$$

We can see that the process is causal as the parameters satisfy the conditions (6.6). We can also check it by calculating the roots of the autoregressive polynomial. These are found by solving the equation

$$\phi(z) = 1 - 0.8z - 0.1z^2 = 0.$$

The discriminant is $\Delta = 0.8^2 + 4 \cdot 0.1 = 1.04$ and the roots are

$$z_1 = \frac{0.8 - \sqrt{1.04}}{2(-0.1)} = 1.09902$$

$$z_2 = \frac{0.8 + \sqrt{1.04}}{2(-0.1)} = -9.09902$$

The roots are outside the interval $[-1, 1]$ and so the process is stationary and causal. Its linear representation is given by (6.5), where

$$\psi_j = \theta_j + \sum_{k=1}^{p} \phi_k \psi_{j-k}.$$

For ARMA(2,1) the only nonzero coefficients ϕ and θ are ϕ_1, ϕ_2 and θ_1, also $p = 2$. Hence, the coefficients ψ are

$$\psi_0 = 1$$
$$\psi_1 = \theta_1 + \phi_1$$
$$\psi_2 = \phi_1\psi_1 + \phi_2\psi_0$$
$$\psi_3 = \phi_1\psi_2 + \phi_2\psi_1$$
$$\psi_4 = \phi_1\psi_3 + \phi_2\psi_2$$
$$\cdots$$
$$\psi_j = \phi_1\psi_{j-1} + \phi_2\psi_{j-2}$$

Note that the set of the above equations can be written as

$$\psi_0 = 1$$
$$\psi_1 = \theta_1 + \phi_1$$
$$\psi_j - \phi_1\psi_{j-1} - \phi_2\psi_{j-2} = 0, \quad \text{for } j > 1.$$

The last equation

$$\psi_j - \phi_1 \psi_{j-1} - \phi_2 \psi_{j-2} = 0 \tag{6.7}$$

is so called **homogeneous difference equation of order 2**, while the first two equations

$$\psi_0 = 1$$
$$\psi_1 = \theta_1 + \phi_1$$

are the **initial conditions**.

The solution to the difference equation (6.8) depends on the solution to the associated polynomial homogeneous equation

$$\phi(z) = 1 - \phi_1 z - \phi_2 z^2 = 0. \tag{6.8}$$

If there are two different roots z_1 and z_2 of (6.9) then the general solution to equation (6.8) is

$$\psi_j = c_1 z_1^{-j} + c_2 z_2^{-j}, \tag{6.9}$$

where c_1 and c_2 depend on initial conditions. This can be verified by direct substitution of ψ_j into (6.8). The initial conditions can be calculated from

$$\psi_0 = c_1 + c_2$$
$$\psi_1 = c_1 z_1^{-1} + c_2 z_2^{-1}$$

When the roots are equal, $z_1 = z_2 (= z_0)$ the general solution to (6.8) is

$$\psi_j = z_0^{-j}(c_1 + c_2 j),$$

where the unknown coefficients c_1 and c_2 can be obtained from the initial conditions and

$$\psi_0 = c_1$$
$$\psi_1 = z_0^{-1}(c_1 + c_2).$$

Here, in the example, we have $\psi_0 = 1$ and $\psi_1 = \theta_1 + \phi_1 = 0.3 + 0.8 = 1.1$ and there are two distinct roots of (6.9). Hence, the initial conditions are

$$c_1 + c_2 = 1$$
$$c_1 z_1^{-1} + c_2 z_2^{-1} = 1.1$$

where $z_1 = 1.099$, $z_2 = -9.099$. The solutions to these equations can then be substituted in (6.10) to find values of the coefficients ψ_j. Here we obtain

$$c_1 = 0.186388$$
$$c_2 = 0.813612,$$

what gives

$$\psi_j = 0.186388 \frac{1}{1.099^j} + 0.813612 \frac{1}{-9.099^j}.$$

6.1.2   Invertibility of ARMA(p,q)

This addresses the problem of uniqueness discussed in Section 4.3.1 which is related to the MA part of the ARMA model. We choose the model which has an infinite autoregressive representation, i.e., is invertible and can be written as

$$Z_t = \frac{1}{\theta(B)}\phi(B)X_t = \pi(B)X_t = \sum_{j=0}^{\infty}\pi_j B^j X_t = \sum_{j=0}^{\infty}\pi_j X_{t-j},$$

where $\sum_{j=0}^{\infty}|\pi_j| < \infty$ and $\pi_0 = 1$. Analogously to the causality condition given in Proposition 6.1 we have the following

Proposition 6.2. The ARMA(p,q) process is invertible, that is there exist constants $\{\pi_j\}$ such that $\sum_{j=0}^{\infty}|\pi_j| < \infty$ and

$$Z_t = \sum_{j=0}^{\infty}\pi_j X_{t-j}, \tag{6.10}$$

if and only if

$$\theta(z) = 1 + \theta_1 z + \ldots + \theta_q z^q = 0 \quad \text{only for } |z| > 1.$$

The coefficients π_j can be determined by solving

$$\pi(z)\theta(z) = \phi(z),$$

where $\pi(z) = \sum_{j=0}^{\infty}\pi_j z^j$.

Example 6.3. **Parameter Redundancy, Causality and Invertibility**

Consider the process

$$X_t - 0.4X_{t-1} - 0.45X_{t-2} = Z_t + Z_{t-1} + 0.25Z_{t-2}.$$

In the operator form it is

$$(1 - 0.4B - 0.45B^2)X_t = (1 + B + 0.25B^2)Z_t.$$

Is this really an ARMA(2,2) process?

We need to check if the polynomials $\phi(z)$ and $\theta(z)$ have common factors. We have

$$\phi(z) = 1 - 0.4z - 0.45z^2 = (1 + 0.5z)(1 - 0.9z),$$

and

$$\theta(z) = 1 + z + 0.25z^2 = (1 + 0.5z)^2.$$

Hence there is one common factor $(1 + 0.5z)$ and the model can be simplified to

$$(1 - 0.9B)X_t = (1 + 0.5B)Z_t,$$

or

$$X_t - 0.9X_{t-1} = Z_t + 0.5Z_{t-1}.$$

It means that in fact it is an ARMA(1,1) model and the parameters ϕ_2 and θ_2 are redundant. That's why in the definition of ARMA(p,q) we have stated that the polynomials $\phi(z)$ and $\theta(z)$ should not have common factors. Then there is no parameter redundancy.

The model is causal because

$$\phi(z) = 1 - 0.9z = 0 \quad \text{when } z = 10/9,$$

which is outside the unit circle. The model is also invertible because

$$\theta(z) = 1 + 0.5z = 0 \quad \text{when } z = -2,$$

which is outside the unit circle too.

To obtain a linear process form of the model we need to calculate the coefficients ψ_j. It can be done from the relation (6.7), which gives the values

$$\psi_j = \theta_j + \sum_{k=1}^{p} \phi_k \psi_{j-k},$$

where $\theta_0 = 1$, $\theta_j = 0$ for $j > q$, and $\psi_j = 0$ for $j < 0$. This gives

$$
\begin{aligned}
\psi_0 &= \theta_0 = 1 \\
\psi_1 &= \theta_1 + \phi_1\psi_0 = \theta_1 + \phi_1 = 0.5 + 0.9 = 1.4 \\
\psi_2 &= \phi_1\psi_1 = \phi_1(\theta_1 + \phi_1) = 0.9 \times 1.4 \\
\psi_3 &= \phi_1\psi_2 = \phi_1^2(\theta_1 + \phi_1) = 0.9^2 \times 1.4 \\
&\cdots \\
\psi_j &= \phi_1\psi_{j-1} = \phi_1^{j-1}(\theta_1 + \phi_1) = 0.9^{j-1} \times 1.4.
\end{aligned}
$$

Hence we can write

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j} = Z_t + 1.4 \sum_{j=1}^{\infty} 0.9^{j-1} Z_{t-j}.$$

Similarly, we can find the invertible representation of the model which is

$$Z_t = \sum_{j=0}^{\infty} \pi_j X_{t-j} = X_t - 1.4 \sum_{j=1}^{\infty} (-0.5)^{j-1} X_{t-j}.$$