

AN EXPLICIT REFINED GAN–GROSS–PRASAD IDENTITY FOR FOURIER–JACOBI PERIODS OF DEGREE 2 SIEGEL CUSP FORMS

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ABSTRACT. We compute the local integrals appearing in the refined Gan–Gross–Prasad conjecture for Fourier–Jacobi periods of Sp_4 in new ramified cases and use this to formulate an explicit conjectural identity relating Petersson norms of degree 2 Siegel cusp forms and associated half-integral weight forms. We note consequences of our identity for the growth of Petersson norms, the size of Fourier coefficients, and non-vanishing of central L -values.

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1. INTRODUCTION

1.1. Main results. Let F be a Siegel cusp form¹ of degree 2 and even weight k for the group $\mathrm{Sp}_4(\mathbb{Z})$. Then F has a Fourier expansion

$$F(Z) = \sum_S a(F, S) e^{2\pi i \mathrm{tr}(SZ)},$$

where the Fourier coefficients $a(F, S)$ are indexed by matrices S of the form

$$(1) \quad S = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}, \quad a, b, c \in \mathbb{Z}, \quad a > 0, \quad \mathrm{disc}(S) := b^2 - 4ac < 0.$$

For each $m > 0$, we can construct a half-integral weight cusp form given by

$$(2) \quad f_m(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z}, \quad \text{where} \quad a(n) = \sum_{\substack{0 \leq r \leq 2m-1 \\ r^2 \equiv -n \pmod{4m}}} a\left(F, \begin{bmatrix} \frac{n+r^2}{4m} & \frac{r}{2} \\ \frac{r}{2} & m \end{bmatrix}\right).$$

It is known that $f_m \in S_{k-\frac{1}{2}}(\Gamma_0(4m))$ (see [12, Theorem 5.6]). The half-integral weight forms f_m are of particular interest for studying F , since their Fourier coefficients, as is evident from (2), are closely related to those of F . Indeed, in the papers [30, 31, 19], the forms f_m were used to prove new results about sign-changes and non-vanishing of fundamental Fourier coefficients of F . It is therefore natural to ask about the relationship between the Petersson norms of f_m and F .

If F is a Hecke eigenform and a *Saito–Kurokawa lift*, and $m = 1$, then Kohnen–Skoruppa [22] (see also [8]) established the identity²

$$(3) \quad \frac{\langle f_1, f_1 \rangle}{\langle F, F \rangle} = \frac{24\pi^k}{\Gamma(k) L(\frac{3}{2}, \pi_0)},$$

where π_0 is the automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ that lifts to the Saito–Kurokawa lift F . Analogous formulas in this Saito–Kurokawa case for $\frac{\langle f_m, f_m \rangle}{\langle F, F \rangle}$ can likely be derived from (3) for all squarefree m , using existing expressions (e.g., [12, Theorem 6.2]) for the m -th Fourier–Jacobi coefficient in terms of the first.

However, if F is *not* a Saito–Kurokawa lift, we are not aware of any previous results on the ratio $\frac{\langle f_m, f_m \rangle}{\langle F, F \rangle}$ for any m . A key result of this paper is an exact conjectural formula for this ratio in terms of a sum of L -values when m is odd and squarefree.

Theorem 1.1 (See Corollary 3.8). *Let F be a Hecke eigenform in $S_k(\mathrm{Sp}_4(\mathbb{Z}))$ and assume that k is even and F is not a Saito–Kurokawa lift. Assume the truth of the refined Gan–Gross–Prasad (GGP) conjecture for Fourier–Jacobi periods on Sp_4 (Conjecture 1 below). Then for each odd squarefree integer m , we have*

$$(4) \quad \frac{\langle f_m, f_m \rangle}{\langle F, F \rangle} = \frac{4\pi^{k+5}}{3(2k-3)\Gamma(k)} \sum_{C|m} \sum_{f \in \mathcal{B}_{2k-2}^{\mathrm{new}}(C)} \frac{L(1/2, \pi_F \times \pi_f)}{L(1, \pi_F, \mathrm{Ad}) L(1, \pi_f, \mathrm{Ad})} \cdot r_{f,m,C}.$$

Above, $\mathcal{B}_{2k-2}^{\mathrm{new}}(C)$ is an orthogonal Hecke basis of the space of holomorphic newforms on the upper half-plane of weight $2k-2$ for $\Gamma_0(C)$, π_F is the automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$ associated to F , π_f is the automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ associated to f , $L(s, \pi_F \times \pi_f)$ denotes the

¹For definitions and background on Siegel cusp forms, see [20] or [27].

²We denote the completed L -functions by $\Lambda(s, \cdot)$ and reserve $L(s, \cdot)$ for the finite part of the L -function. L -functions in this paper are normalized so that the global functional equation for the completed L -function relates $s \mapsto 1-s$.

degree 10 L -function attached to the tensor product of the degree-five standard parameter of π_F and the degree-two parameter of π_f , and the quantity $r_{f,m,C}$ is given by

$$r_{f,m,C} = \prod_{p \mid \frac{m}{C}} \frac{2(\alpha_p^{-1/2} + \alpha_p^{1/2})^2(\beta_p^{-1/2} + \beta_p^{1/2})^2}{(p+1)(1 + \delta_p p^{-1/2})(1 + \delta_p^{-1} p^{-1/2})} \prod_{p \nmid C} \frac{2}{p+1}$$

if the local Atkin–Lehner eigenvalue for f equals 1 at all primes dividing C and $r_{f,m,C} = 0$ otherwise. Here, for each prime not dividing C , $\alpha_p, \alpha_p^{-1}, \beta_p, \beta_p^{-1}$ are the Satake parameters of π_F at p , and δ_p, δ_p^{-1} are the Satake parameters of π_f at p .

Remark 1.2. We note that $L(1, \pi_F, \text{Ad})$ is not zero by Theorem 5.2.1 of [28].

Remark 1.3. We can regard (4) as a generalization of (3). In the case of Saito–Kurokawa lifts, $L(s, \pi_F, \text{Ad})$ has a pole at $s = 1$, and an analog of (4) for $m = 1$ would reduce to a single nonzero term corresponding to the unique f (which lifts to F) for which $L(s, \pi_F \times \pi_f)$ also has a pole at $s = 1/2$. Simplifying the resulting expression gives us (3) up to a constant.

Theorem 1.1 implies an essentially optimal upper bound for $\langle f_m, f_m \rangle$ under GRH.

Corollary 1.4. Let F, m be as in Theorem 1.1. Assume the refined GGP conjecture for Fourier–Jacobi periods on Sp_4 and also assume the Generalized Riemann Hypothesis (GRH) for the L -functions appearing in Theorem 1.1. Then

$$\langle f_m, f_m \rangle \ll_{F,\epsilon} m^\epsilon.$$

Proof. Under GRH, we can bound each $\frac{L(1/2, \pi_F \times \pi_f)}{L(1, \pi_F, \text{Ad})L(1, \pi_f, \text{Ad})}$ by $\ll_{k,\epsilon} m^\epsilon$. The cardinality of $\mathcal{B}_{2k-2}^{\text{new}}(C)$ is $\ll_k m$ and each factor $r_{f,m,C}$ is bounded by $m^{-1+\epsilon}$. The proof now follows from (4). Note that the dependencies on k are absorbed by those on F . $\square$

The above results have applications to upper bounds on Fourier coefficients of F and to the non-vanishing of central L -values, see Section 1.4.

1.2. The refined Gan–Gross–Prasad conjectures for Fourier–Jacobi periods. We now explain how Theorem 1.1 fits into the framework of the refined GGP conjectures for Fourier–Jacobi periods on symplectic groups formulated by Xue in [39]. Let G denote the algebraic group Sp_4 , and let H be the Heisenberg group:

$$H := \left\{ \begin{bmatrix} 1 & & \mu & \\ & 1 & \kappa & \\ \lambda & 1 & 1 & -\lambda \\ & & 1 & \end{bmatrix} \right\} \subset \text{Sp}_4.$$

We consider SL_2 as a subgroup of G via

$$(5) \quad \text{SL}_2 \ni \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a & b & & \\ c & 1 & & \\ & & d & \\ & & & 1 \end{bmatrix} \in \text{Sp}_4.$$

Let $\widetilde{\text{SL}}_2$ denote the metaplectic double cover of SL_2 . Define $J := \text{SL}_2 \cdot H \subset \text{Sp}_4$ and $\widetilde{J} := \widetilde{\text{SL}}_2 \ltimes H$.

Let $\mathbb{A}$ denote the ring of adeles of $\mathbb{Q}$. We fix the standard additive character $\psi = \prod_v \psi_v : \mathbb{Q} \backslash \mathbb{A} \rightarrow \mathbb{C}^\times$, characterized by

$$(6) \quad \psi_\infty(x) = e^{2\pi i x}, \quad \text{cond}(\psi_p) = \mathbb{Z}_p \quad (p < \infty).$$

We equip $\mathbb{A}$ with the Haar measure that is self-dual with respect to ψ . Concretely, this is the product of the local self-dual Haar measures: Lebesgue measure dx_∞ on $\mathbb{R}$, and, for every prime p , the Haar measure dx_p on $\mathbb{Q}_p$ normalized by $\text{vol}_{dx_p}(\mathbb{Z}_p) = 1$.

For each $m \in \mathbb{Q}^\times$, let ψ^m be the non-trivial additive character $\mathbb{Q} \backslash \mathbb{A}$ defined by $\psi^m(x) := \psi(mx)$, and let ω_{ψ^m} denote the Schrödinger–Weil representation of $\widetilde{J}(\mathbb{A}) = \widetilde{\mathrm{SL}}_2(\mathbb{A}) \ltimes H(\mathbb{A})$ realized on the Schwartz space $\mathcal{S}(\mathbb{A})$. For $\phi \in \mathcal{S}(\mathbb{A})$ and $r \in \widetilde{J}(\mathbb{A})$, we define the theta function

$$\Theta_{\psi^m}(r, \phi) := \sum_{x \in \mathbb{Q}} (\omega_{\psi^m}(r)\phi)(x).$$

Let (π, V_π) be an irreducible, cuspidal, automorphic representation of $G(\mathbb{A})$ and (σ, V_σ) be an irreducible cuspidal genuine automorphic representation of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$. For $\Psi \in V_\pi$, $\Lambda \in V_\sigma$, $\phi \in \mathcal{S}(\mathbb{A})$, and $m \in \mathbb{Q}^\times$, define the global Fourier–Jacobi period

$$(7) \quad \mathcal{FJ}_{\psi^m}(\Psi, \Lambda, \phi) := \int_{\mathrm{SL}_2(\mathbb{Q}) \backslash \mathrm{SL}_2(\mathbb{A})} \int_{H(\mathbb{Q}) \backslash H(\mathbb{A})} \Psi(ng) \Lambda(g) \overline{\Theta_{\psi^m}(ng, \phi)} \, dn \, dg.$$

Above, we fix the measure dg to be the global *Tamagawa measure* and we normalize the Haar measure on $H(\mathbb{Q}) \backslash H(\mathbb{A})$ so that the total volume is 1. Fix local Haar measures dn_v on $H(\mathbb{Q}_v)$ (resp. dg_v on $\mathrm{SL}_2(\mathbb{Q}_v)$) so that the subgroups $H(\mathbb{Z}_p)$ and $\mathrm{SL}_2(\mathbb{Z}_p)$ have volume one at all finite primes p . At the archimedean place, we choose the measure on $H(\mathbb{R})$ to be the usual Lebesgue measure. Note that $dn = \prod_v dn_v$. We normalize the Haar measure dg_∞ on $\mathrm{SL}_2(\mathbb{R})$ by $dg_\infty = y^{-2} dx dy dk_\infty$, where $g_\infty = \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} \begin{bmatrix} y^{1/2} & \\ & y^{-1/2} \end{bmatrix} k_\infty$ is the Iwasawa decomposition, and dk_∞ is the measure on $\mathrm{SO}(2)$ so that it has volume 1. Define the constant C_G by $dg = C_G \prod_v dg_v$.

By the tensor product theorem, $\pi \cong \otimes' \pi_v$ with irreducible, admissible representations π_v of $G(\mathbb{Q}_v)$, and $\sigma \cong \otimes' \sigma_v$ with irreducible, admissible representations σ_v of $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_v)$. Assume that Ψ corresponds to a pure tensor $\otimes \Psi_v$, and that Λ corresponds to a pure tensor $\otimes \Lambda_v$. Assume further that $\phi = \otimes \phi_v$ with local Schwartz functions $\phi_v \in \mathcal{S}(\mathbb{Q}_v)$. For each place v , fix a $G(\mathbb{Q}_v)$ -invariant Hermitian inner product on the space of π_v , and an $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_v)$ -invariant Hermitian inner product on σ_v . For each place v , let us define the local integral

$$\alpha_v(\Psi_v, \Lambda_v, \phi_v; \psi_v^m) = \int_{\mathrm{SL}_2(\mathbb{Q}_v)} \int_{H(\mathbb{Q}_v)} \langle \pi_v(ng) \Psi_v, \Psi_v \rangle \langle \sigma_v(g) \Lambda_v, \Lambda_v \rangle \overline{\langle \omega_{\psi_v^m}(ng) \phi_v, \phi_v \rangle} \, dn \, dg.$$

Define the normalized local integrals

$$\alpha_v^\#(\Psi_v, \Lambda_v, \phi_v; \psi_v^m) := \frac{\alpha_v(\Psi_v, \Lambda_v, \phi_v; \psi_v^m)}{\langle \Psi_v, \Psi_v \rangle \langle \Lambda_v, \Lambda_v \rangle \langle \phi_v, \phi_v \rangle} \times \left(\zeta_v(2) \zeta_v(4) \frac{L_{\psi_v^m}(\frac{1}{2}, \pi_v \times \sigma_v)}{L(1, \pi_v, \mathrm{Ad}) L_{\psi_v^m}(1, \sigma_v, \mathrm{Ad})} \right)^{-1}.$$

Above, $L_{\psi_v^m}(\cdot)$ denotes the local L -functions defined using the Langlands parameters of the representations π_v and σ_v , which in the case of σ_v also depend on the character ψ_v^m (see Section 11 of [13]).

By work of Xue [39] we have $\alpha_v^\#(\Psi_v, \Lambda_v, \phi_v; \psi_v^m) = 1$ for almost all places v . Inspired by work of Ichino–Ikeda [17], Xue [39, 40] conjectured the following.

Conjecture 1 (see (1.2) of [40]). *Assume that π and σ are tempered, and generic almost everywhere. Then*

$$(8) \quad \frac{|\mathcal{FJ}_{\psi^m}(\Psi, \Lambda, \phi)|^2}{\langle \Psi, \Psi \rangle \langle \Lambda, \Lambda \rangle \langle \phi, \phi \rangle} = 2^{-\beta} \xi(2)^2 \xi(4) C_G \frac{\Lambda_{\psi^m}(1/2, \pi \times \sigma)}{\Lambda(1, \pi, \mathrm{Ad}) \Lambda_{\psi^m}(1, \sigma, \mathrm{Ad})} \times \prod_v \alpha_v^\#(\Psi_v, \Lambda_v, \phi_v; \psi_v^m).$$

Here the positive integer β is 1 if π is of general type (so that its lift to $\mathrm{GL}_5(\mathbb{A})$ is cuspidal) and 2 if π is endoscopic. The function $\xi(s)$ above is the completed Riemann zeta function.

Remark 1.5. Suppose that the global Waldspurger lifting of σ to $\mathrm{PGL}_2(\mathbb{A})$ with respect to ψ^m exists, and let $\pi_0 = \mathrm{Wald}_{\psi^m}(\sigma)$ denote this lift. Then we have $L_{\psi^m}(s, \pi \times \sigma) = L(s, \pi \times \pi_0)$, and $L_{\psi^m}(s, \sigma, \mathrm{Ad}) = L(s, \pi_0, \mathrm{Ad})$.

Remark 1.6. *In the setup in [39, 40], the local Haar measures on $H(\mathbb{Q}_v)$ are taken to be self-dual with respect to ψ_v^m . In contrast, above, we chose the coordinate measures to be self-dual with respect to the standard character ψ_v . This does not affect Conjecture 1. Indeed, changing ψ_v to ψ_v^m multiplies the self-dual measure on $H(\mathbb{Q}_v) \simeq \mathbb{Q}_v^3$ by $|m|_v^{3/2}$, whose global product is one, and so the right side of (8) is unchanged.*

1.3. Computing local integrals. We now briefly describe our main local result. Fix the following purely local data, where for convenience we have dropped all subscripts.

- i) A non-archimedean local field F of characteristic zero, with ring of integers $\mathfrak{o}$, maximal ideal $\mathfrak{p}$, uniformizer ϖ , and residue field of odd cardinality q . Let v be the normalized valuation on F , so that $v(\varpi) = 1$. We fix an additive character ψ of F with conductor $\mathfrak{o}$, and normalize measures as in Section 2.
- ii) An element $m \in F^\times$, and the character $\psi^m(x) = \psi(mx)$. Put

$$\phi = \mathbf{1}_{\mathfrak{o}}, \quad \phi^{(m)}(x) = \phi(mx).$$

- iii) An irreducible, unramified, tempered representation π of $\mathrm{GSp}_4(F)$ with trivial central character³, together with a spherical vector v_0 . Then π is of the form $\chi_1 \times \chi_2 \rtimes \chi_0$ with unitary characters χ_0, χ_1, χ_2 satisfying $\chi_0^2 \chi_1 \chi_2 = 1$. Let

$$(9) \quad \alpha = \chi_1(\varpi), \quad \beta = \chi_2(\varpi).$$

- iv) An irreducible, genuine, unitary, tempered representation σ of $\widetilde{\mathrm{SL}}_2(F)$, which we assume is not an even Weil representation. Let $\sigma' = m^{-1} \cdot \sigma$ be the twist defined in Section 2.
- v) The Heisenberg group $H(F)$ and the Schrödinger-Weil representation ω_{ψ^m} of $\widetilde{\mathrm{SL}}_2(F) \ltimes H(F)$, realized on $\mathcal{S}(F)$.

Let $\Gamma^0(m) \subset \mathrm{SL}_2(\mathfrak{o})$ be the subgroup whose upper-right entry lies in $m\mathfrak{o}$, and let χ_m be the quadratic character of $F^\times$ given by $\chi_m(x) = (x, m)$, where $(\cdot, \cdot)$ is the local Hilbert symbol. We also use χ_m to denote the resulting character of $\Gamma^0(m)$ via its action on the top left entry.

Let $B_{\sigma'}(m)$ be an orthogonal basis of the subspace of σ' transforming under $\Gamma^0(m)$ by χ_m (see (20)), and put

$$\alpha^\#(\pi, \sigma; m) = \sum_{\Lambda' \in B_{\sigma'}(m)} \alpha^\#(v_0, \Lambda', \phi^{(m)}; \psi^m),$$

where the normalized local integral on the right was defined above. The quantity $\alpha^\#(\pi, \sigma; m)$ is independent of the choice of the orthogonal basis and of the normalizations of the invariant inner products.

When $v(m) = 0$, the above quantity can be nonzero only when σ is unramified, and in that case Xue proved that $\alpha^\#(\pi, \sigma; m) = 1$ [39]. Our main local result is an exact formula for the case $v(m) = 1$.

The condition $V_{\sigma'}(\Gamma^0(m), \chi_m) \neq 0$ implies that σ has a $\Gamma_0(\mathfrak{p})$ -fixed vector. Under the assumptions above, there are two cases:

- i) $\sigma \simeq \tilde{\pi}(\chi)$, a metaplectic principal series representation, where χ is a unitary, unramified character of $F^\times$. In this case the relevant fixed subspace has dimension two.
- ii) $\sigma \simeq \tilde{\sigma}_\xi$, a metaplectic special representation, with $\xi \in \mathfrak{o}^\times$. In this case the relevant fixed subspace has dimension one.

³Although Conjecture 1 involves a representation π_v of $\mathrm{Sp}_4(F)$, we may work here with a representation of $\mathrm{GSp}_4(F)$ whose restriction to $\mathrm{Sp}_4(F)$ contains π_v : the normalized matrix coefficients and the L -functions occurring in the local integral are all unchanged; see also the analogous global discussion in Section 3.3.

Our main local result, Theorem 2.2, gives the following explicit values. If $\sigma \simeq \tilde{\pi}(\chi)$ and $\delta = \chi(\varpi)$, then

$$\alpha^\#(\pi, \sigma; m) = \frac{2}{q+1} \frac{(\alpha^{-1/2} + \alpha^{1/2})^2 (\beta^{-1/2} + \beta^{1/2})^2}{(1 + \delta q^{-1/2})(1 + \delta^{-1} q^{-1/2})}.$$

If $\sigma \simeq \tilde{\sigma}_\xi$ and $\varepsilon_\sigma = (\varpi, \xi) \in \{\pm 1\}$, then

$$\alpha^\#(\pi, \sigma; m) = \begin{cases} 0 & \varepsilon_\sigma = 1, \\ \frac{2}{q+1} & \varepsilon_\sigma = -1. \end{cases}$$

The detailed calculations occupy Section 2. We briefly indicate the key steps. A twisting argument first replaces ψ^m and $\phi^{(m)}$ by ψ and ϕ , at the cost of conjugating the spherical matrix coefficient of π . The resulting period is then rewritten as an integral over the Jacobi group involving the $K_0^J(\mathfrak{p})$ -fixed matrix coefficients of $\sigma \otimes \omega_{\psi^{-1}}$. In the unramified principal series case there are two diagonal terms. In the special case, the fixed vector is obtained from the kernel of a standard intertwining operator on the full induced representation, with its invariant inner product obtained from the derivative of that operator.

In both cases, the period reduces to essentially the same integral (see (74) and Proposition 2.10), after introducing a spherical damping factor $T^{\text{ht}(g)}$ to ensure absolute convergence and justify the required changes of variables. Splitting the central variable according to its valuation shows that three pieces contribute. We evaluate these pieces in Sections 2.8–2.10 by combining the Cartan decomposition criterion of [21, Proposition 3.2] with the Macdonald formula for the spherical matrix coefficient of π , following the general strategy of [11].⁴ The resulting expressions are rational in the Satake parameters and T , and the limit $T \rightarrow 1$ recovers the original period. After inserting the local L -factors, they simplify to the formulas above.

1.4. The global identity and some consequences. Returning to Conjecture 1, we set $\Psi = \Psi_F$ to be the adelization of a Hecke eigenform $F \in S_k(\text{Sp}_4(\mathbb{Z}))$ (as before, we assume k is even and F is not a Saito–Kurokawa lift). We interpret the adelic global Fourier–Jacobi period, defined in (7), classically in terms of Petersson inner products of half-integral weight forms. Precisely, for a Hecke eigenform $h \in S_{k-\frac{1}{2}}^+(\Gamma_0(4m))$, we choose the global data so that the left-hand side of (8) becomes a multiple of

$$\frac{|\langle f_m, h \rangle|^2}{\langle F, F \rangle \langle h, h \rangle}.$$

On the other hand, the local result described in the previous subsection, together with archimedean computations, allows us to write down the right hand side of (8) explicitly.

The Shimura–Waldspurger correspondence, refined by Kohnen, relates h to a Hecke eigenform f of weight $2k - 2$ and level m ; precisely, $\pi_f = \text{Wald}_\psi(\sigma_h)$ is the Waldspurger lifting of the contragredient of the automorphic representation σ_h generated by the adelization of h . This leads to our main global theorem (Theorem 3.7), which expresses $\frac{|\langle f_m, h \rangle|^2}{\langle F, F \rangle \langle h, h \rangle}$ exactly in terms of L -values and local parameters. In the special case that h is a *newform* of level $4m$ (so that f is a newform of weight $2k - 2$ and level m), the identity is as follows:

$$(10) \quad \frac{|\langle f_m, h \rangle|^2}{\langle h, h \rangle \langle F, F \rangle} = \frac{4\pi^{k+5}}{3(2k-3)\Gamma(k)} \cdot \frac{L(1/2, \pi_F \times \pi_f)}{L(1, \pi_F, \text{Ad})L(1, \pi_f, \text{Ad})} \cdot \prod_{p|m} r_{f,p}$$

where $r_{f,p} = \frac{2}{p+1}$ if the local Atkin–Lehner eigenvalue of f at p equals 1 and $r_{f,p} = 0$ if this eigenvalue equals -1 . A similar formula as above holds for oldforms h . By summing over an

⁴These computations are also formalized in the Lean 4 proof assistant; see Remark 2.17.

orthogonal Hecke basis of the relevant half-integral weight space and applying Parseval, we deduce Theorem 1.1.

We noted in Corollary 1.4 above that our results imply strong upper bounds on $\langle f_m, f_m \rangle$ under GRH. It turns out that they also imply strong upper bounds on the size of the Fourier coefficients $a(F, S)$. For simplicity, assume that S is fundamental, i.e., $\text{disc}(S)$ is a fundamental discriminant. A famously difficult conjecture of Resnikoff and Saldana [29] predicts that

$$(11) \quad |a(F, S)| \ll_{F, \epsilon} (\det S)^{\frac{k}{2} - \frac{3}{4} + \epsilon}.$$

Assuming GRH, it was proved in [19, Theorem C] that

$$(12) \quad |a(F, S)| \ll_{F, \epsilon} \frac{(\det S)^{\frac{k}{2} - \frac{1}{2}}}{(\log |(\det S)|)^{\frac{1}{8} - \epsilon}}.$$

This bound represents the limit of the current technology and assumes GRH; yet it still does not get us to the bound (11), making it clear how deep that conjecture is.

The best unconditional bound toward (11) is due to Kohnen [23] who proved that $|a(F, S)| \ll_{F, \epsilon} (\det S)^{\frac{k}{2} - \frac{13}{36} + \epsilon}$. This follows from the following bound proved by Kohnen in the same paper:

$$(13) \quad |a(F, S)| \ll_{F, \epsilon} (\min S)^{5/18 + \epsilon} (\det S)^{\frac{k}{2} - \frac{1}{2} + \epsilon}$$

where $\min S$ is the smallest positive integer represented by S .

Since $\text{disc}(S)$ is a fundamental discriminant, S is primitive, and therefore represents infinitely many primes [18, Theorem 1 (i)]. If S represents 1, then put $\min_{\text{pr}} S = 1$, and otherwise, let $\min_{\text{pr}} S$ denote the smallest odd prime represented by S . Assuming GRH and Conjecture 1, we prove the following bound:

$$(14) \quad |a(F, S)| \ll_{F, \epsilon} (\min_{\text{pr}} S)^{1/2 + \epsilon} (\det S)^{\frac{k}{2} - \frac{3}{4} + \epsilon}$$

This result follows from our main global theorem and Waldspurger’s theorem. In fact, we prove a more precise version of (14) where the implied constant is explicit; see (244). Note that (14) is stronger than (12) when $\min_{\text{pr}} S \leq (\det S)^{1/2 - \delta}$ for some $\delta > 0$.

Theorem 1.1 also implies (assuming Conjecture 1) a non-vanishing result for certain central L -values: for each odd squarefree m such that $f_m \neq 0$, at least one of the $L(1/2, \pi_F \times \pi_f)$ on the right hand side of (4) must be non-zero. There are infinitely many such m ; indeed, a positive density of odd primes have this property (see Remark 3.12). We do not, however, attempt to obtain an effective bound for the smallest such m .⁵

The Fourier coefficients of Siegel Hecke cusp forms of degree $n \geq 2$ are mysterious objects. For $n = 2$, this paper provides one approach to studying their analytic properties, based on the refined GGP identity for Fourier–Jacobi periods.⁶ Another approach, developed in [11, 10], uses the refined GGP identity for Bessel periods. However, the Bessel-period approach relies on the accidental isomorphism $\text{PGSp}_4 \simeq \text{SO}_5$, so it is special to degree 2. By contrast, the Fourier–Jacobi framework is available for Sp_{2n} in every degree. It is thus natural to expect higher-rank analogues of the conjectural identity studied in this paper; working out their precise formulations and implications would be an important direction for future work.

⁵A recent paper of Manickam [24] claims to prove that $f_1 \neq 0$. Unfortunately, the argument appears to contain a gap, as we explain in Remark 3.13.

⁶While this identity remains conjectural at the time of writing, a proof of the analogous identity for unitary groups has recently been announced by Boisseau–Lu–Xue [7].

1.5. Declaration on the use of AI. During the preparation of Sections 2.4–2.6, OpenAI’s GPT-5.5, GPT-5.6 Sol, and GPT-6 Astra were used interactively to explore ideas and assist with symbolic computations. The models were also used elsewhere in the paper to identify relevant literature, check calculations, help detect errors, and proofread the manuscript. The formalization of part of Section 2 in Lean 4, subsequently registered in Palomar [1], was done with the help of GPT-6 Astra in Codex. All AI-assisted material in this paper was carefully reviewed, verified, and revised by the authors, who take full responsibility for the contents of the paper.

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2. LOCAL CALCULATIONS

2.1. Notation. Let F be a non-archimedean local field of characteristic zero and odd residue characteristic, $\mathfrak{o}$ the ring of integers of F , and $\mathfrak{p}$ the maximal ideal of $\mathfrak{o}$. Let $\varpi \in \mathfrak{o}$ be a fixed generator of $\mathfrak{p}$, and $q = \#(\mathfrak{o}/\mathfrak{p})$ be the cardinality of the residue class field. The normalized valuation on F is denoted by v , and the normalized absolute value by $|\cdot|$. We fix a non-trivial character ψ of F with conductor $\mathfrak{o}$ once and for all. We let $(\cdot, \cdot): F^\times \times F^\times \rightarrow \{\pm 1\}$ denote the Hilbert symbol. For $m \in F^\times$, we let χ_m be the quadratic character of $F^\times$ given by $\chi_m(x) = (x, m)$.

The Haar measure on F is normalized such that $\mathfrak{o}$ has volume 1, and the Haar measure on $F^\times$ is normalized such that $\mathfrak{o}^\times$ has volume $1 - q^{-1}$.

For an ideal $\mathfrak{m}$ of $\mathfrak{o}$, let $\Gamma^0(\mathfrak{m})$ denote the subgroup of $\mathrm{SL}_2(F)$ consisting of the elements in $\mathrm{SL}_2(\mathfrak{o})$ whose top right entry is in $\mathfrak{m}$, and let $\Gamma_0(\mathfrak{m})$ be the subgroup consisting of elements with bottom left entry in $\mathfrak{m}$. If $m \in \mathfrak{o}$, we sometimes write $\Gamma^0(m)$ instead of $\Gamma^0(m\mathfrak{o})$, and $\Gamma_0(m)$ instead of $\Gamma_0(m\mathfrak{o})$.

Let $m \in F^\times$, and set $\psi^m(x) := \psi(mx)$. Note that the conductor of ψ^m is $\mathfrak{p}^{-v(m)}$. For now, m is arbitrary, but we will assume later that $v(m) = 1$.

Throughout Section 2, we use a different matrix realization of the symplectic, Jacobi and Heisenberg groups than in the introduction. Precisely, let

$$\mathrm{GSp}_4(F) = \{g \in \mathrm{GL}_4(F) : {}^t g J' g = \lambda(g) J', \lambda(g) \in F^\times\}, \quad J' = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix}.$$

Put $K = \mathrm{GSp}_4(\mathfrak{o})$. We let $\mathrm{Sp}_4(F)$ denote the subgroup of $\mathrm{GSp}_4(F)$ for which $\lambda(g) = 1$. We embed $\mathrm{SL}_2(F)$ inside $\mathrm{Sp}_4(F)$ by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} 1 & & & \\ & a & b & \\ & c & d & \\ & & & 1 \end{bmatrix}.$$

We let w be the image of $\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}$, i.e.,

$$(15) \quad w = \begin{bmatrix} 1 & & & \\ & -1 & 1 & \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

Define the *Heisenberg group*

$$(16) \quad H(F) = \left\{ \begin{bmatrix} 1 & x & y & z \\ & 1 & y & \\ & & 1 & -x \\ & & & 1 \end{bmatrix} : x, y, z \in F \right\}.$$

The semidirect product $J(F) := \mathrm{SL}_2(F) \ltimes H(F)$ is called the *Jacobi group*. We normalize the Haar measure on $\mathrm{SL}_2(F)$ so that $\mathrm{SL}_2(\mathfrak{o})$ has volume 1. Then, for any integrable function φ on $\mathrm{SL}_2(F)$,

$$(17) \quad \int_{\mathrm{SL}_2(F)} \varphi(A) dA = \frac{1}{1 - q^{-1}} \int_{F^\times} \int_F \int_{\mathrm{SL}_2(\mathfrak{o})} \varphi\left(\begin{bmatrix} a & \\ & a^{-1} \end{bmatrix} \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} \kappa\right) d\kappa dx d^\times a.$$

The Haar measure on F leads to a Haar measure on $H(F)$. The Jacobi group carries the product measure.

2.2. The metaplectic group and the Jacobi group. Let $\widetilde{\mathrm{SL}}_2(F)$ be the metaplectic group. As a set, it is $\mathrm{SL}_2(F) \times \{\pm 1\}$, with multiplication defined by the modified Kubota cocycle, as in [14] or [37]. For any $y \in F^\times$ and $g \in \mathrm{SL}_2(F)$, define

$$(18) \quad g^y = \begin{bmatrix} y & \\ & 1 \end{bmatrix} g \begin{bmatrix} y^{-1} & \\ & 1 \end{bmatrix}.$$

This pulls back uniquely to an automorphism $g \mapsto g^y$ of $\widetilde{\mathrm{SL}}_2(F)$. If (σ, V_σ) is a representation of $\widetilde{\mathrm{SL}}_2(F)$, and if $m \in F^\times$, then the twisted representation $m \cdot \sigma$ is the representation of $\widetilde{\mathrm{SL}}_2(F)$ on the same space V_σ given by

$$(19) \quad (m \cdot \sigma)(g) = \sigma(g^m).$$

Smooth and admissible representations are defined as usual, and the twisting operation preserves these attributes. A representation σ of $\widetilde{\mathrm{SL}}_2(F)$ is called *genuine* if the element $(1, -1)$ acts by multiplication by -1 . This property is also preserved by twisting.

Since we assume the residual characteristic of F is odd, the direct product $\mathrm{SL}_2(\mathfrak{o}) \times \{\pm 1\}$ is a subgroup of $\widetilde{\mathrm{SL}}_2(F)$; see pages 17–19 of [14]. In particular, $\Gamma^0(m) \times \{\pm 1\}$ and $\Gamma_0(m) \times \{\pm 1\}$ are subgroups of $\widetilde{\mathrm{SL}}_2(F)$.

Let (σ, V_σ) be a smooth, genuine representation of $\widetilde{\mathrm{SL}}_2(F)$. If Γ is a congruence subgroup of $\mathrm{SL}_2(\mathfrak{o})$, and if χ is a character of Γ , we denote by $V_\sigma(\Gamma, \chi)$ the space of $v \in V_\sigma$ for which

$$(20) \quad \sigma(\gamma, \varepsilon)v = \varepsilon\chi(\gamma)v \quad \text{for all } \gamma \in \Gamma, \varepsilon \in \{\pm 1\}.$$

If $v(m) > 0$, and if we view a character χ of $F^\times$ that is trivial on $1 + m\mathfrak{o}$ as a character of $\Gamma_0(m)$ (or $\Gamma^0(m)$) via $\chi(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \chi(a)$, then it is easy to see that

$$(21) \quad V_\sigma(\Gamma_0(m), \chi) = V_{m^{-1}\sigma}(\Gamma^0(m), \chi\chi_m).$$

Let ω_ψ be the Weil representation for the character ψ and the one-dimensional quadratic form $q(x) = x^2$; see page 223 of [37]. Note that ω_{ψ^m} is the Weil representation for the character ψ and the quadratic form $q(x) = mx^2$. The space of ω_ψ is $\mathcal{S}(F)$, and the subspaces of even and odd Schwartz functions are invariant and irreducible. If $\mathcal{S}(F)$ is equipped with the inner product $\langle \phi, \phi' \rangle = \int_F \phi(x)\overline{\phi'(x)} dx$, then ω_ψ becomes a unitary representation. It has an extension, also unitary, to $\mathrm{SL}_2(F) \ltimes H(F)$, called the *Schrödinger–Weil representation*; see [6, Section 2.5]. Sometimes it is more convenient to view ω_ψ as a projective representation of $J(F) = \mathrm{SL}_2(F) \ltimes H(F)$. For later use we note that if we define $\phi^{(m)}(x) = \phi(mx)$ for $m \in F^\times$, then

$$(22) \quad \omega_{\psi^m} \left(\begin{bmatrix} 1 & xm^{-1} & y & zm^{-1} \\ & 1 & y & \\ & & 1 & -xm^{-1} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & a & bm & \\ & cm^{-1} & d & \\ & & & 1 \end{bmatrix} \right) (\phi^{(m)}) = \left(\omega_\psi \left(\begin{bmatrix} 1 & x & y & z \\ & 1 & y & \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & a & b & \\ & c & d & \\ & & & 1 \end{bmatrix} \right) \phi \right)^{(m)}$$

for $\phi \in \mathcal{S}(F)$.

For a character χ of $F^\times$, we define the metaplectic induced representation $\tilde{\pi}(\chi)$ as on page 225 of [37]. Note that the definition depends on ψ , which we consider fixed and suppress from the notation. If $\chi^2 \neq |\cdot|^{\pm 1}$, then $\tilde{\pi}(\chi)$ is irreducible, and we call it a *principal series representation*. If $\chi = |\cdot|^{1/2}\chi_\xi$ for some $\xi \in F^\times$, then $\tilde{\pi}(\chi)$ contains a unique irreducible subrepresentation $\tilde{\sigma}_\xi$, called a *special representation*. The quotient $\tilde{\pi}(\chi)/\tilde{\sigma}_\xi$ is isomorphic to an even Weil representation.

Lemma 2.1. *Let (σ, V_σ) be an irreducible, admissible, genuine representation of $\widetilde{\mathrm{SL}}_2(F)$.*

- i) Assume that $V_\sigma(\mathrm{SL}_2(\mathfrak{o}), 1) \neq 0$. Then either σ is an even Weil representation, or $\sigma = \tilde{\pi}(\chi)$ with an unramified character χ of $F^\times$ for which $\chi^2 \neq |\cdot|^{\pm 1}$. In the latter case*

$$(23) \quad \dim V_\sigma(\mathrm{SL}_2(\mathfrak{o}), 1) = 1, \quad \dim V_\sigma(\Gamma_0(\mathfrak{p}), 1) = \dim V_\sigma(\Gamma^0(\mathfrak{p}), 1) = 2.$$

- ii) Assume that $V_\sigma(\mathrm{SL}_2(\mathfrak{o}), 1) = 0$ and $V_\sigma(\Gamma_0(\mathfrak{p}), 1) \neq 0$. Then $\sigma = \tilde{\sigma}_\xi$ with $\xi \in \mathfrak{o}^\times$ and*

$$(24) \quad \dim V_\sigma(\Gamma_0(\mathfrak{p}), 1) = 1.$$

Proof. See [6, Section 5.3]. □

The “Borel subgroup” of the Jacobi group $J(F)$ is

$$(25) \quad B^J(F) = \left\{ \begin{bmatrix} 1 & & & \\ & a & b & \\ & & a^{-1} & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & y & z \\ & 1 & y \\ & & 1 & \\ & & & 1 \end{bmatrix} : a \in F^\times, b, y, z \in F \right\}.$$

For a character η of $F^\times$, we let $\pi^J(\eta)$ be the principal series representation of $J(F)$ consisting of smooth functions $f: J(F) \rightarrow \mathbb{C}$ which are compactly supported modulo $B^J(F)$ (this is no longer automatic) with the transformation property

$$(26) \quad f\left(\begin{bmatrix} 1 & * & z \\ & a & * \\ & & a^{-1} \\ & & & 1 \end{bmatrix} g\right) = \psi^{-1}(z)\eta(a)|a|^{3/2}f(g)$$

for $a \in F^\times$, $z \in F$ and $g \in J(F)$. If $\sigma = \tilde{\pi}(\eta)$ is a principal series representation induced from a character η of $F^\times$, then it follows from [6, Theorem 5.4.2] that

$$(27) \quad \tilde{\pi}(\eta) \otimes \omega_{\psi^{-1}} \cong \pi^J(\eta).$$

If χ is a unitary unramified character of $F^\times$, then one can show with the help of [9, Lemma 2.6.1] that an invariant Hermitian inner product on the standard space of $\pi^J(\chi)$ is given by

$$(28) \quad \langle f, f' \rangle = \int_{\mathrm{SL}_2(\mathfrak{o})} \int_F f\left(\begin{bmatrix} 1 & u & \\ & 1 & \\ & & 1 & -u \\ & & & 1 \end{bmatrix} \kappa\right) \overline{f'\left(\begin{bmatrix} 1 & u & \\ & 1 & \\ & & 1 & -u \\ & & & 1 \end{bmatrix} \kappa\right)} du d\kappa.$$

The integral (28) is convergent because the f, f' are compactly supported modulo $B^J(F)$. In general, when η is not unitary, the formula (28) does not give an invariant inner product on $\pi^J(\eta)$, even when the latter is unitary.

Recall that if $\eta = |\cdot|^{1/2}\chi_\xi$, then $\tilde{\pi}(\eta)$ contains the special representation $\sigma = \tilde{\sigma}_\xi$ as a subrepresentation. In this case it follows from (27) that $\pi^J(\eta)$ contains a corresponding (irreducible) subrepresentation π_ξ^J , which we also call a special representation.

One can prove (see [6, Theorem 2.6.2]) that every irreducible, admissible representation of $J(F)$ with central character ψ^{-1} is of the form $\sigma \otimes \omega_{\psi^{-1}}$ for some irreducible, admissible representation σ of $\widetilde{\mathrm{SL}}_2(F)$. Here, we view both σ and $\omega_{\psi^{-1}}$ as projective representations of $J(F)$; their cocycles cancel each other out in the tensor product so as to produce a true representation.

2.3. The local Fourier–Jacobi integral and the main local theorem. Let π be an irreducible, unramified, tempered representation of $\mathrm{GSp}_4(F)$ with trivial central character. Then there exist unramified, unitary characters χ_0, χ_1, χ_2 of $F^\times$ such that $\pi = \chi_1 \times \chi_2 \rtimes \chi_0$; see [32]. Let $\alpha = \chi_1(\varpi)$, $\beta = \chi_2(\varpi)$, $\gamma = \chi_0(\varpi)$. Since the central character of π is trivial, $\alpha\beta\gamma^2 = 1$. Then the degree-5 L -factor attached to π is given by

$$L(s, \pi, \rho_5) = ((1 - q^{-s})(1 - \alpha q^{-s})(1 - \alpha^{-1} q^{-s})(1 - \beta q^{-s})(1 - \beta^{-1} q^{-s}))^{-1}.$$

Let v_0 be a (unique up to multiples) spherical (i.e., non-zero K -fixed) vector in π . Fix an invariant inner product on the space of π . Let $\Phi_0: \mathrm{GSp}_4(F) \rightarrow \mathbb{C}$ be the normalized spherical matrix coefficient, defined by

$$(29) \quad \Phi_0(g) = \frac{\langle \pi(g)v_0, v_0 \rangle}{\langle v_0, v_0 \rangle}$$

for $g \in \mathrm{GSp}_4(F)$. Let m be a non-zero element of $\mathfrak{o}$. Define

$$\phi = \mathbf{1}_{\mathfrak{o}}, \quad \phi^{(m)}(x) = \phi(mx) = \begin{cases} 1 & \text{if } x \in m^{-1}\mathfrak{o}, \\ 0 & \text{otherwise.} \end{cases}$$

Let (σ, V_σ) be an irreducible, admissible, genuine, unitary, tempered representation of $\widetilde{\mathrm{SL}}_2(F)$. Fix an invariant inner product $\langle \cdot, \cdot \rangle$ on V_σ . Let $\sigma' = m^{-1} \cdot \sigma$ be the twisted representation; see (19). Then $\langle \cdot, \cdot \rangle$ is also an invariant inner product for σ' . For Λ' in $V_{\sigma'} = V_\sigma$, consider the local quantity

$$(30) \quad \alpha(v_0, \Lambda', \phi^{(m)}; \psi^m) = \int_{\mathrm{SL}_2(F)} \int_{H(F)} \Phi_0(n g) \langle \sigma'(g) \Lambda', \Lambda' \rangle \overline{\langle \omega_{\psi^m}(n g) \phi^{(m)}, \phi^{(m)} \rangle} dn dg.$$

Let $B_{\sigma'}(m)$ be an orthogonal basis for $V_{\sigma'}(\Gamma^0(m), \chi_m)$; see (20). (If m is a unit, we view χ_m as the trivial character of $\Gamma^0(m) = \mathrm{SL}_2(\mathfrak{o})$.) We define

$$(31) \quad \alpha(\pi, \sigma; m) = \sum_{\Lambda' \in B_{\sigma'}(m)} \frac{\alpha(v_0, \Lambda', \phi^{(m)}; \psi^m)}{\langle \Lambda', \Lambda' \rangle \langle \phi^{(m)}, \phi^{(m)} \rangle}.$$

Note that $\alpha(\pi, \sigma; m)$ does not depend on the choice of the orthogonal basis or on the normalization of the inner products. Define the normalized local factor

$$(32) \quad \begin{aligned} \alpha^\#(\pi, \sigma; m) &= (1 - q^{-2})(1 - q^{-4}) \frac{L(1, \pi, \mathrm{Ad}) L_{\psi^m}(1, \sigma', \mathrm{Ad})}{L_{\psi^m}(\frac{1}{2}, \pi \times \sigma')} \alpha(\pi, \sigma; m) \\ &= (1 - q^{-2})(1 - q^{-4}) \frac{L(1, \pi, \mathrm{Ad}) L_\psi(1, \sigma, \mathrm{Ad})}{L_\psi(\frac{1}{2}, \pi \times \sigma)} \alpha(\pi, \sigma; m). \end{aligned}$$

Here the local L -factors involving a metaplectic representation are defined using its additive-character-dependent local Langlands parameter (see Section 11 of [13]). What it means is that

$$(33) \quad L_\psi(s, \sigma, \mathrm{Ad}) = L(s, \tau, \mathrm{Ad}), \quad L_\psi(s, \pi \times \sigma) = L(s, \pi \times \tau),$$

where τ is the Waldspurger lift of σ (see [37, Section VI.2]). Note that τ is an irreducible, admissible representation of $\mathrm{GL}_2(F)$ with trivial central character depending on the choice of ψ .

In the case $v(m) = 0$, it is easy to see that $\alpha^\#(\pi, \sigma; m)$ is non-zero only if σ' , and hence σ , are unramified principal series representations, as otherwise $V_{\sigma'}(\Gamma^0(m), \chi_m) = 0$; see Lemma 2.1. In this case it was proved by Xue [39] (assuming that q is odd) that $\alpha^\#(\pi, \sigma; m) = 1$.

So the natural next question is to compute $\alpha^\#(\pi, \sigma; m)$ when $v(m) = 1$. In this case, for $\alpha^\#(\pi, \sigma; m)$ to be non-zero, $B_{\sigma'}(m)$ has to be non-empty, i.e., $V_{\sigma'}(\Gamma^0(m), \chi_m) \neq 0$. By (21), this is equivalent to $V_\sigma(\Gamma_0(m), 1) \neq 0$. We assume further that σ is not an even Weil representation. Recall also that σ is tempered. Hence by Lemma 2.1, σ is of one of the following types:

- i) (Unramified principal series representation) $\sigma \simeq \tilde{\pi}(\chi)$, where χ is a unitary unramified character of $F^\times$ and $\chi^2 \neq |\cdot|^{\pm 1}$. In this case, $B_{\sigma'}(m)$ has two elements.
- ii) (Special representation) $\sigma \simeq \tilde{\sigma}_\xi$ with $\xi \in \mathfrak{o}^\times$. In this case, $B_{\sigma'}(m)$ is a singleton set.

Our main local result computes the local quantity $\alpha^\#(\pi, \sigma; m)$ in the above two cases. Recall that we assume that q is odd.

Theorem 2.2. *Let $v(m) = 1$, and let the quantity $\alpha^\#(\pi, \sigma; m)$ be defined as in (32).*

- i) *Suppose that $\sigma \simeq \tilde{\pi}(\chi)$, where χ is a unitary unramified character of $F^\times$, and put $\delta = \chi(\varpi)$. Then*

$$\alpha^\#(\pi, \sigma; m) = \frac{2}{q+1} \cdot \frac{(\alpha^{-1/2} + \alpha^{1/2})^2 (\beta^{-1/2} + \beta^{1/2})^2}{(1 + \delta q^{-1/2})(1 + \delta^{-1} q^{-1/2})}.$$

- ii) *Suppose that $\sigma \simeq \tilde{\sigma}_\xi$ is a special representation with $\xi \in \mathfrak{o}^\times$, and put $\varepsilon_\sigma = (\varpi, \xi) \in \{\pm 1\}$. Then*

$$\alpha^\#(\pi, \sigma; m) = \begin{cases} 0, & \varepsilon_\sigma = 1, \\ \frac{2}{q+1}, & \varepsilon_\sigma = -1. \end{cases}$$

2.4. Initial reductions. Throughout the rest of Section 2 we assume that $v(m) = 1$. Thus $m\mathfrak{o} = \mathfrak{p}$ and $|m| = q^{-1}$. In this subsection, we reduce $\alpha(\pi, \sigma; m)$ to a variant where the representations of the metaplectic and Jacobi groups are simpler, but at the cost of shifting the matrix coefficient of GSp_4 . For any $t \in F^\times$ define

$$(34) \quad e_t = \mathrm{diag}(t, 1, t, 1) \in \mathrm{GSp}_4(F).$$

For any $h \in \mathrm{GSp}_4(F)$ define

$$(35) \quad \Phi_0^m(h) = \Phi_0(e_m^{-1}he_m) = \frac{\langle \pi(h)v_0^m, v_0^m \rangle}{\langle v_0^m, v_0^m \rangle}, \text{ where } v_0^m = \pi(e_m)v_0.$$

By [39, Proposition 2.2.1], the integrals (30) occurring in (31) are absolutely convergent. The same holds when we replace $(\Phi_0, \psi^m, \phi^{(m)})$ in the integral by (Φ_0^m, ψ, ϕ) . The next lemma records a more precise version of this that will be useful later.

Lemma 2.3. *Fix a tempered representation σ of $\widetilde{\mathrm{SL}}_2(F)$ and vectors $\Lambda_1, \Lambda_2 \in V_\sigma$. Then*

$$\int_{\mathrm{SL}_2(F)} \int_{H(F)} |\Phi_0^m(ng)| |\langle \sigma(g)\Lambda_1, \Lambda_2 \rangle| |\langle \omega_\psi(ng)\phi, \phi \rangle| dn dg$$

is absolutely convergent and admits an absolutely integrable majorant independent of the unitary unramified Satake parameters of π .

Proof. Let Ξ_G be the Harish–Chandra spherical function of $\mathrm{GSp}_4(F)$. The spherical integral formula gives $|\Phi_0^m(h)| \leq \Xi_G(e_m^{-1}he_m)$ uniformly in π . Replacing $|\Phi_0^m(ng)|$ by this bound gives an integrable function by applying [39, Proposition 2.2.1] to the tempered principal series induced from the trivial character, and its translated spherical vector. This is the required majorant. $\square$

Lemma 2.4. *Let $B_\sigma(\mathfrak{p})$ be an orthogonal basis for $V_\sigma(\Gamma_0(\mathfrak{p}), 1)$. Note that this basis has 2 elements if σ is an unramified principal series representation, and has 1 element if σ is special. Then we have*

$$(36) \quad \alpha(\pi, \sigma; m) = q^2 \sum_{\Lambda \in B_\sigma(\mathfrak{p})} \frac{\alpha(v_0^m, \Lambda, \phi; \psi)}{\langle \Lambda, \Lambda \rangle \langle \phi, \phi \rangle},$$

where

$$(37) \quad \alpha(v_0^m, \Lambda, \phi; \psi) = \int_{\mathrm{SL}_2(F)} \int_{H(F)} \Phi_0^m(ng) \langle \sigma(g)\Lambda, \Lambda \rangle \overline{\langle \omega_\psi(ng)\phi, \phi \rangle} dn dg.$$

Proof. By (21), we may take $B_{\sigma'}(m) = B_\sigma(\mathfrak{p})$ in (31). Now perform the variable transformation $ng \rightarrow e_m^{-1}nge_m$ in the defining integral (30). Observing (22), we easily obtain the asserted formula. $\square$

By Lemma 2.4, we are reduced to computing

$$\alpha(\pi, \sigma; m) = q^2 \sum_{\Lambda \in B_\sigma(\mathfrak{p})} \int_{\mathrm{SL}_2(F)} \int_{H(F)} \Phi_0^m(ng) \frac{\langle \sigma(g)\Lambda, \Lambda \rangle \overline{\langle \omega_\psi(ng)\phi, \phi \rangle}}{\langle \Lambda, \Lambda \rangle \langle \phi, \phi \rangle} dn dg,$$

where Φ_0^m is defined in (35). We define $K^J = J(\mathfrak{o})$. Define the subgroup $K_0^J(\mathfrak{p})$ of K^J by

$$(38) \quad K_0^J(\mathfrak{p}) = K^J \cap \begin{bmatrix} 1 & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \Gamma_0(\mathfrak{p}) \ltimes H(\mathfrak{o}).$$

Also define the subgroup $K_1^J(\mathfrak{p})$ of $J(F)$ by

$$(39) \quad K_1^J(\mathfrak{p}) = J(F) \cap \begin{bmatrix} 1 & \mathfrak{p} & \mathfrak{o} & \mathfrak{p} \\ \mathfrak{o} & \mathfrak{p}^{-1} & \mathfrak{o} & \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{p} & \\ 0 & 0 & 0 & 1 \end{bmatrix} = J(F) \cap e_m K e_m^{-1},$$

where e_m was defined in (34). Note that the function Φ_0^m is bi-invariant under the subgroup $K_1^J(\mathfrak{p})$.

Let Π^J be the unitary representation of $J(F)$ defined by

$$(40) \quad \Pi^J = \sigma \otimes \omega_{\psi^{-1}}.$$

We let $(\Pi^J)^{K_0^J(\mathfrak{p})}$ denote the subspace of Π^J that is fixed by $K_0^J(\mathfrak{p})$.

Lemma 2.5. *Let B_Π be an orthogonal basis of $(\Pi^J)^{K_0^J(\mathfrak{p})}$. Then*

$$\alpha(\pi, \sigma; m) = q^2 \int_{J(F)} \Phi_0^m(j) \sum_{f \in B_\Pi} \frac{\langle \Pi^J(j)f, f \rangle}{\langle f, f \rangle} dj.$$

Proof. Since q is odd and ψ has conductor $\mathfrak{o}$, the vector $\phi = \mathbf{1}_{\mathfrak{o}}$ is fixed by $\omega_{\psi^{-1}}(\mathrm{SL}_2(\mathfrak{o}))$ and by $\omega_{\psi^{-1}}(H(\mathfrak{o}))$. Moreover, it spans the line $\mathcal{S}(F)^{H(\mathfrak{o})}$; see [6, Lemma 6.3.2]. It follows that

$$(\Pi^J)^{K_0^J(\mathfrak{p})} = V_\sigma^{\Gamma_0(\mathfrak{p})} \otimes \mathbb{C}\phi.$$

Since $v(m) = 1$, we have $\Gamma_0(m) = \Gamma_0(\mathfrak{p})$. Hence, if $B_\sigma(\mathfrak{p})$ is the orthogonal basis of the $\Gamma_0(\mathfrak{p})$ -fixed subspace of σ occurring in Lemma 2.4, then $\{\Lambda \otimes \phi : \Lambda \in B_\sigma(\mathfrak{p})\}$ is an orthogonal basis of $(\Pi^J)^{K_0^J(\mathfrak{p})}$. Now writing $j = ng$, we have

$$\frac{\langle \Pi^J(ng)(\Lambda \otimes \phi), \Lambda \otimes \phi \rangle}{\langle \Lambda \otimes \phi, \Lambda \otimes \phi \rangle} = \frac{\langle \sigma(g)\Lambda, \Lambda \rangle}{\langle \Lambda, \Lambda \rangle} \cdot \frac{\langle \omega_{\psi^{-1}}(ng)\phi, \phi \rangle}{\langle \phi, \phi \rangle}.$$

Using the above and $\langle \omega_{\psi^{-1}}(ng)\phi, \phi \rangle = \overline{\langle \omega_\psi(ng)\phi, \phi \rangle}$ we get

$$\begin{aligned} & \int_{J(F)} \Phi_0^m(j) \sum_{f \in B_\Pi} \frac{\langle \Pi^J(j)f, f \rangle}{\langle f, f \rangle} dj \\ &= \sum_{\Lambda \in B_\sigma(\mathfrak{p})} \int_{\mathrm{SL}_2(F)} \int_{H(F)} \Phi_0^m(ng) \frac{\langle \sigma(g)\Lambda, \Lambda \rangle}{\langle \Lambda, \Lambda \rangle} \frac{\overline{\langle \omega_\psi(ng)\phi, \phi \rangle}}{\langle \phi, \phi \rangle} dn dg. \end{aligned}$$

By (36), the right-hand side is $q^{-2}\alpha(\pi, \sigma; m)$, as claimed. $\square$

2.5. Further reductions. We now analyze the two cases separately. We first consider the case that $\sigma = \tilde{\pi}(\chi)$ is an unramified principal series representation with χ unitary. Let f_0 be the unramified, i.e. K^J -invariant, vector in $\pi^J(\chi)$, normalized by $f_0(1) = 1$. By the considerations in [6, Section 6.3],

$$(41) \quad f_0\left(\begin{bmatrix} 1 & * & z \\ a & * & * \\ a^{-1} & * & * \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & u & & \\ & 1 & & \\ & & 1 & -u \\ & & & 1 \end{bmatrix} \kappa\right) = \begin{cases} \psi^{-1}(z)\chi(a)|a|^{3/2} & \text{if } u \in \mathfrak{o}, \\ 0 & \text{if } u \notin \mathfrak{o}, \end{cases}$$

for $\kappa \in K^J$. Using (28), it follows that $\langle f_0, f_0 \rangle = 1$ and

$$(42) \quad \langle \Pi^J(g)f_0, f_0 \rangle = \int_{\mathrm{SL}_2(\mathfrak{o})} \int_{\mathfrak{o}} f_0\left(\begin{bmatrix} 1 & u & & \\ & 1 & & \\ & & 1 & -u \\ & & & 1 \end{bmatrix} \kappa g\right) du d\kappa$$

for $g \in J(F)$.

Let $w = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \in \mathrm{SL}_2(\mathfrak{o})$, identified with the element of K^J defined in (15). Let $B_1(\mathfrak{o})$ denote the upper triangular subgroup of $\mathrm{SL}_2(\mathfrak{o})$. Let f_1 be the function in $\pi^J(\chi)$ supported on $B^J(F)K_0^J(\mathfrak{p})$, normalized by $f_1(1) = 1$, and let f_2 be the function supported on $B^J(F)wK_0^J(\mathfrak{p})$, normalized by $f_2(w) = 1$. Then

$$f_0 = f_1 + f_2,$$

and

(43)

$$\langle f_1, f_1 \rangle = \text{vol}(B_1(\mathfrak{o})\Gamma_0(\mathfrak{p})) = \frac{1}{q+1}, \quad \langle f_2, f_2 \rangle = \text{vol}(B_1(\mathfrak{o})w\Gamma_0(\mathfrak{p})) = \frac{q}{q+1}, \quad \langle f_1, f_2 \rangle = 0.$$

Put $g_1 = 1$ and $g_2 = w$. Generalizing (42), we have, for $i \in \{1, 2\}$,

$$(44) \quad \langle \Pi^J(g)f_i, f_i \rangle = \int_{B_1(\mathfrak{o})g_i\Gamma_0(\mathfrak{p})} \int_{\mathfrak{o}} f_i \left(\begin{bmatrix} 1 & u & & \\ & 1 & & \\ & & 1 & -u \\ & & & 1 \end{bmatrix} \kappa g \right) du d\kappa$$

for $g \in J(F)$. Define for $i = 1, 2$,

$$(45) \quad \Phi_i^J(g) = \frac{\langle \Pi^J(g)f_i, f_i \rangle}{\langle f_i, f_i \rangle},$$

and

$$(46) \quad I(\Phi_0^m, \Phi_i^J) = \int_{J(F)} \Phi_0^m(g) \Phi_i^J(g) dg.$$

Therefore, taking $B_\Pi = \{f_1, f_2\}$ in Lemma 2.5, we obtain in this case:

$$(47) \quad \alpha(\pi, \sigma; m) = q^2 (I(\Phi_0^m, \Phi_1^J) + I(\Phi_0^m, \Phi_2^J)),$$

and by (44),

$$(48) \quad I(\Phi_0^m, \Phi_i^J) = \frac{1}{\text{vol}(B_1(\mathfrak{o})g_i\Gamma_0(\mathfrak{p}))} \int_{J(F)} \Phi_0^m(g) \left(\int_{B_1(\mathfrak{o})g_i\Gamma_0(\mathfrak{p})} \int_{\mathfrak{o}} f_i \left(\begin{bmatrix} 1 & u & & \\ & 1 & & \\ & & 1 & -u \\ & & & 1 \end{bmatrix} \kappa g \right) du d\kappa \right) dg.$$

We now treat the case where σ is a special representation, i.e., $\sigma = \tilde{\sigma}_\xi$ with $\xi \in \mathfrak{o}^\times$. The formula (28) is not an invariant Hermitian inner product on the special representation. To obtain the correct inner product, first define

$$(49) \quad \eta_{s_0} = (\cdot, \xi) | \cdot |^{s_0}, \quad \pi_{s_0}^J = \pi^J(\eta_{s_0}).$$

Thus $\eta = \eta_{1/2}$. For $f \in \pi_{s_0}^J$ and $f' \in \pi_{-s_0}^J$, define the pairing

$$(50) \quad \mathcal{P}_{s_0}(f, f') = \int_{\text{SL}_2(\mathfrak{o})} \int_F f \left(\begin{bmatrix} 1 & u & & \\ & 1 & & \\ & & 1 & -u \\ & & & 1 \end{bmatrix} \kappa \right) \overline{f' \left(\begin{bmatrix} 1 & u & & \\ & 1 & & \\ & & 1 & -u \\ & & & 1 \end{bmatrix} \kappa \right)} du d\kappa.$$

(When $s_0 \in i\mathbb{R}$, this reduces to the invariant inner product on the associated principal series representation defined earlier.) Let

$$A_{s_0}: \pi_{s_0}^J \longrightarrow \pi_{-s_0}^J$$

be the intertwining operator defined by

$$(51) \quad (A_{s_0}f)(g) = \int_F \int_F f \left(w \begin{bmatrix} 1 & y & & \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix} g \right) dx dy, \quad w = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix},$$

for $\text{Re}(s_0) > 0$ and by meromorphic continuation for all s_0 . For $i = 1, 2$, let $f_{i,s_0} \in \pi_{s_0}^J$ be the two $K_0^J(\mathfrak{p})$ -fixed vectors with the same support and normalization as above, and write

$$f_i = f_{i,1/2}, \quad f_i^\vee = f_{i,-1/2}.$$

Relative to the bases $(f_{1,-s_0}, f_{2,-s_0})$ and (f_{1,s_0}, f_{2,s_0}) , it is easy to check that the matrix of A_{s_0} on the $K_0^J(\mathfrak{p})$ -fixed vectors is

$$(52) \quad M_{s_0} = \begin{bmatrix} \frac{q^{-2s_0}(1-q^{-1})}{1-q^{-2s_0}} & 1 \\ q^{-1} & \frac{1-q^{-1}}{1-q^{-2s_0}} \end{bmatrix}.$$

In particular,

$$(53) \quad M_{1/2} = \begin{bmatrix} q^{-1} & 1 \\ q^{-1} & 1 \end{bmatrix},$$

and therefore

$$A_{1/2}(c_1 f_1 + c_2 f_2) = 0 \iff q^{-1}c_1 + c_2 = 0.$$

Thus the $K_0^J(\mathfrak{p})$ -fixed line in the special representation is generated by

$$(54) \quad f_{\text{sp}} = f_1 - q^{-1}f_2.$$

To define the invariant inner product on the special representation, put

$$\dot{A}_{1/2} := -\frac{q-1}{2\log q} \frac{d}{ds_0} \Big|_{s_0=1/2} A_{s_0}.$$

The scalar is chosen only for convenience and has no effect on normalized matrix coefficients. Note that $\dot{A}_{1/2}f_1 = f_1^\vee$ and $\dot{A}_{1/2}f_2 = f_2^\vee$, so that

$$(55) \quad \dot{A}_{1/2}f_{\text{sp}} = f_1^\vee - q^{-1}f_2^\vee =: f_{\text{sp}}^\vee.$$

We define the invariant Hermitian inner product on the special representation by

$$(56) \quad \langle f, h \rangle := \mathcal{P}_{1/2}(f, \dot{A}_{1/2}h), \quad f, h \in \ker(A_{1/2}) \simeq \Pi^J.$$

Remark 2.6. *The fact that (56) defines a $J(F)$ -invariant inner product on $\ker(A_{1/2}) \simeq \tilde{\sigma}_\xi \otimes \omega_{\psi^{-1}}$ follows by a standard limiting argument for complementary series inner products; cf. [15, Chapter 1, formulas (300)–(310)].*

For $0 \leq s_0 \leq \frac{1}{2}$, we obtain the following immediately from the support of f_{1,s_0} and f_{2,s_0} :

$$(57) \quad \begin{aligned} \mathcal{P}_{s_0}(f_{1,s_0}, f_{1,-s_0}) &= \text{vol}(B_1(\mathfrak{o})\Gamma_0(\mathfrak{p})) = \frac{1}{q+1}, \\ \mathcal{P}_{s_0}(f_{2,s_0}, f_{2,-s_0}) &= \text{vol}(B_1(\mathfrak{o})w\Gamma_0(\mathfrak{p})) = \frac{q}{q+1}, \\ \mathcal{P}_{s_0}(f_{1,s_0}, f_{2,-s_0}) &= \mathcal{P}_{s_0}(f_{2,s_0}, f_{1,-s_0}) = 0. \end{aligned}$$

Consequently, if we define

$$(58) \quad f_{\text{sp},s_0} := f_{1,s_0} - q^{-1}f_{2,s_0} \in \pi_{s_0}^J, \quad f_{\text{sp},s_0}^\vee := f_{1,-s_0} - q^{-1}f_{2,-s_0} \in \pi_{-s_0}^J,$$

then

$$\mathcal{P}_{s_0}(f_{\text{sp},s_0}, f_{\text{sp},s_0}^\vee) = \frac{1}{q}.$$

Putting $s_0 = \frac{1}{2}$, we get

$$(59) \quad \langle f_{\text{sp}}, f_{\text{sp}} \rangle = \mathcal{P}_{1/2}(f_1 - q^{-1}f_2, f_1^\vee - q^{-1}f_2^\vee) = \frac{1}{q}.$$

Now Lemma 2.5 gives

$$(60) \quad \alpha(\pi, \sigma; m) = q^2 I(\Phi_0^m, \Phi_{\text{sp}}^J),$$

where

$$\Phi_{\text{sp}}^J(g) = \frac{\langle \Pi^J(g)f_{\text{sp}}, f_{\text{sp}} \rangle}{\langle f_{\text{sp}}, f_{\text{sp}} \rangle} = \frac{\mathcal{P}_{1/2}(\Pi^J(g)f_{\text{sp}}, f_{\text{sp}}^\vee)}{\mathcal{P}_{1/2}(f_{\text{sp}}, f_{\text{sp}}^\vee)}, \quad I(\Phi_0^m, \Phi_{\text{sp}}^J) = \int_{J(F)} \Phi_0^m(g) \Phi_{\text{sp}}^J(g) dg.$$

For $0 \leq s_0 \leq \frac{1}{2}$, define

$$\Phi_{\text{sp},s_0}^J(g) := \frac{\mathcal{P}_{s_0}(\pi_{s_0}^J(g)f_{\text{sp},s_0}, f_{\text{sp},s_0}^\vee)}{\mathcal{P}_{s_0}(f_{\text{sp},s_0}, f_{\text{sp},s_0}^\vee)}, \quad I(\Phi_0^m, \Phi_{\text{sp},s_0}^J) := \int_{J(F)} \Phi_0^m(g) \Phi_{\text{sp},s_0}^J(g) dg.$$

Thus $\Phi_{\text{sp},1/2}^J = \Phi_{\text{sp}}^J$.

We would like to expand $I(\Phi_0^m, \Phi_{\text{sp},s_0}^J)$ into diagonal and cross terms using (58), but before doing so we need to introduce a damping function that makes those auxiliary integrals convergent at $s_0 = \frac{1}{2}$. For $g \in \text{GSp}_4(F)$ put

$$(61) \quad \begin{aligned} \text{ht}(g) &= \log_q \left(\frac{\max_{i,j} |g_{ij}|}{|\lambda(g)|^{1/2}} \right), \\ \Phi_{0,T}(g) &= T^{\text{ht}(g)} \Phi_0(g), \quad \Phi_{0,T}^m(g) = \Phi_{0,T}(e_m^{-1} g e_m), \quad 0 < T \leq 1. \end{aligned}$$

Whenever $g \in \text{GSp}_4(F)$ is such that $\text{ht}(g)$ is a non-negative integer, we extend the same definition of $\Phi_{0,T}(g)$ to all complex T with $|T| \leq 1$, with the convention $0^0 = 1$. In particular, this extension applies to every $g \in \text{Sp}_4(F)$, since $\text{ht}(g)$ is then a non-negative integer.

Lemma 2.7. *i) For $i, j \in \{1, 2\}$, the integrals*

$$\int_{J(F)} \Phi_0^m(g) \mathcal{P}_{s_0}(\pi_{s_0}^J(g) f_{i,s_0}, f_{j,-s_0}) dg, \quad \int_{J(F)} \Phi_{0,T}^m(g) \mathcal{P}_{s_0}(\pi_{s_0}^J(g) f_{i,s_0}, f_{j,-s_0}) dg$$

are absolutely convergent for $0 \leq s_0 < 1/2$ and for $0 \leq s_0 \leq 1/2$, $|T| < 1$, respectively. For each fixed s_0 , the latter integral is holomorphic in T on $|T| < 1$.

ii) If $\sigma = \tilde{\pi}(\chi)$ with χ unitary and unramified, then, for $i = 1, 2$,

$$\lim_{T \uparrow 1} \int_{J(F)} \Phi_{0,T}^m(g) \Phi_i^J(g) dg = \int_{J(F)} \Phi_0^m(g) \Phi_i^J(g) dg = I(\Phi_0^m, \Phi_i^J).$$

If $\sigma = \tilde{\sigma}_\xi$ with $\xi \in \mathfrak{o}^\times$, then

$$\lim_{T \uparrow 1} \int_{J(F)} \Phi_{0,T}^m(g) \Phi_{\text{sp}}^J(g) dg = \int_{J(F)} \Phi_0^m(g) \Phi_{\text{sp}}^J(g) dg = I(\Phi_0^m, \Phi_{\text{sp}}^J).$$

The integrals above involving $\Phi_{0,T}^m$ are absolutely convergent for $|T| \leq 1$ and holomorphic in T for $|T| < 1$.

iii) For $0 < |T| < q^{-6}$, $0 \leq s_0 \leq 1/2$ and $i, j \in \{1, 2\}$,

$$\int_{J(F)} \int_{B_1(\mathfrak{o}) g_j \Gamma_0(\mathfrak{p})} \int_{\mathfrak{o}} \left| \Phi_{0,T}^m(g) f_{i,s_0} \left(\begin{bmatrix} 1 & u & & \\ & 1 & & \\ & & 1 & -u \\ & & & 1 \end{bmatrix} \kappa g \right) \right| du d\kappa dg < \infty.$$

The same holds with f_{i,s_0} replaced by the previously chosen normalized $K_0^J(\mathfrak{p})$ -fixed section $f_i \in \pi^J(\chi)$, where χ is unitary and unramified. For fixed T in this range, the bounds are uniform in s_0 and χ , respectively.

Proof. We have the standard spherical estimate

$$(62) \quad \begin{aligned} |\Phi_0(g)| &\leq \Xi_G(g) \ll (1 + A + B)^d q^{-2A-B}, \\ g &\in K \text{diag}(\varpi^A, \varpi^B, \varpi^{-B}, \varpi^{-A}) K, \quad A, B \in \mathbb{Z}, \quad A \geq B \geq 0. \end{aligned}$$

where Ξ_G is the Harish–Chandra spherical function and $d > 0$ is fixed (and independent of the unitary unramified Satake parameters of π). Standard estimates on metaplectic matrix coefficients, together with (62), bound the absolute integral with kernel $\Phi_{0,T}^m$ in i), uniformly for $|T| \leq t$, by

$$(63) \quad C_t \sum_{n \geq 0} (1+n)^D (t q^{s_0-1/2})^n, \quad 0 < t \leq 1,$$

where D and C_t are some fixed positive constants. This gives i), including holomorphy by local uniform domination. Part ii) follows from Lemma 2.3 and dominated convergence and holomorphic parameter integration. For iii), put $N = \text{ht}(e_m^{-1} g e_m)$. The Iwasawa formula (17) shows that the

volume of the corresponding set is $O((1+N)q^{4N})$, while the normalized sections are $O(q^{2N})$ there, uniformly in the stated parameters. Since $|\Phi_0| \leq 1$, the absolute integral is bounded by

$$C \sum_{N \geq 0} (1+N)^D (|T|q^6)^N,$$

which converges in the asserted range. $\square$

For $|T| < 1$, and a measurable function Ψ on $J(F)$, define the “damped integral”

$$I(\Phi_{0,T}^m, \Psi) := \int_{J(F)} \Phi_{0,T}^m(g) \Psi(g) dg$$

whenever it converges absolutely.

For $0 \leq s_0 \leq \frac{1}{2}$ and $|T| < 1$, define the two auxiliary cross terms

$$(64) \quad \mathcal{C}_{21,s_0}(\Phi_{0,T}^m) = \int_{J(F)} \Phi_{0,T}^m(g) \mathcal{P}_{s_0}(\pi_{s_0}^J(g) f_{2,s_0}, f_{1,-s_0}) dg$$

and

$$(65) \quad \mathcal{C}_{12,s_0}(\Phi_{0,T}^m) = \int_{J(F)} \Phi_{0,T}^m(g) \mathcal{P}_{s_0}(\pi_{s_0}^J(g) f_{1,s_0}, f_{2,-s_0}) dg.$$

Lemma 2.7 i) gives absolute convergence of the above integrals. Using (50) and the support of $f_{1,-s_0}$ and $f_{2,-s_0}$, the first cross term can be expressed as

$$(66) \quad \mathcal{C}_{21,s_0}(\Phi_{0,T}^m) = \int_{J(F)} \Phi_{0,T}^m(g) \left(\int_{B_1(\mathfrak{o})\Gamma_0(\mathfrak{p})} \int_{\mathfrak{o}} f_{2,s_0} \left(\begin{bmatrix} 1 & u \\ & 1 & -u \\ & & 1 \end{bmatrix} \kappa g \right) du d\kappa \right) dg.$$

Similarly, the second cross term is

$$(67) \quad \mathcal{C}_{12,s_0}(\Phi_{0,T}^m) = \int_{J(F)} \Phi_{0,T}^m(g) \left(\int_{B_1(\mathfrak{o})w\Gamma_0(\mathfrak{p})} \int_{\mathfrak{o}} f_{1,s_0} \left(\begin{bmatrix} 1 & u \\ & 1 & -u \\ & & 1 \end{bmatrix} \kappa g \right) du d\kappa \right) dg.$$

For the diagonal terms, define, for $i = 1, 2$ and $0 \leq s_0 \leq \frac{1}{2}$, analogously to (45),

$$(68) \quad \Phi_{i,s_0}^J(g) := \frac{\mathcal{P}_{s_0}(\pi_{s_0}^J(g) f_{i,s_0}, f_{i,-s_0})}{\mathcal{P}_{s_0}(f_{i,s_0}, f_{i,-s_0})} = \frac{\mathcal{P}_{s_0}(\pi_{s_0}^J(g) f_{i,s_0}, f_{i,-s_0})}{\text{vol}(B_1(\mathfrak{o})g_i\Gamma_0(\mathfrak{p}))}.$$

Their damped integrals for $|T| < 1$ are given by

$$(69) \quad I(\Phi_{0,T}^m, \Phi_{i,s_0}^J) = \int_{J(F)} \Phi_{0,T}^m(g) \Phi_{i,s_0}^J(g) dg.$$

These integrals converge by Lemma 2.7 i). These are given by the same explicit integral formula as in (48), with f_i replaced by f_{i,s_0} and Φ_0^m by $\Phi_{0,T}^m$. For $0 \leq s_0 \leq \frac{1}{2}$, expanding $f_{\text{sp},s_0} = f_{1,s_0} - q^{-1}f_{2,s_0}$ and $f_{\text{sp},s_0}^\vee = f_{1,-s_0} - q^{-1}f_{2,-s_0}$, and using $\mathcal{P}_{s_0}(f_{\text{sp},s_0}, f_{\text{sp},s_0}^\vee) = q^{-1}$, gives

$$(70) \quad \begin{aligned} I(\Phi_{0,T}^m, \Phi_{\text{sp},s_0}^J) &= \frac{q}{q+1} I(\Phi_{0,T}^m, \Phi_{1,s_0}^J) + \frac{1}{q+1} I(\Phi_{0,T}^m, \Phi_{2,s_0}^J) \\ &\quad - \mathcal{C}_{21,s_0}(\Phi_{0,T}^m) - \mathcal{C}_{12,s_0}(\Phi_{0,T}^m). \end{aligned}$$

Note that by our notation we have $I(\Phi_{0,T}^m, \Phi_{\text{sp},1/2}^J) = I(\Phi_{0,T}^m, \Phi_{\text{sp}}^J)$. Lemma 2.7 ii) gives

$$(71) \quad I(\Phi_0^m, \Phi_{\text{sp}}^J) = \lim_{T \uparrow 1} I(\Phi_{0,T}^m, \Phi_{\text{sp}}^J).$$

Therefore (70) and (60) give

$$(72) \quad \alpha(\pi, \sigma; m) = \lim_{T \uparrow 1} \left(\frac{q^3}{q+1} I(\Phi_{0,T}^m, \Phi_{1,1/2}^J) + \frac{q^2}{q+1} I(\Phi_{0,T}^m, \Phi_{2,1/2}^J) - q^2 (\mathcal{C}_{21,1/2}(\Phi_{0,T}^m) + \mathcal{C}_{12,1/2}(\Phi_{0,T}^m)) \right).$$

The limit is taken on the complete combination. The identification of this limit with an explicit rational expression, including at exceptional unitary Satake parameters of π , is proved in Section 2.11.

2.6. The key simplification. The goal of this subsection is to simplify (47) and (72) into more explicit and computable forms.

For $a \in F^\times$ and $x, y, z \in F$, put

$$(73) \quad g(a; x, y, z) = \begin{bmatrix} 1 & a & & \\ & a^{-1} & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & y & z \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

Throughout the definitions and the next two lemmas, let $\eta = \eta_0 |\cdot|^{s_0}$, where η_0 is a unitary unramified character of $F^\times$ and $0 \leq s_0 \leq \frac{1}{2}$. We allow complex T with $|T| < 1$, and also $T = 1$ when $s_0 < \frac{1}{2}$, using the convention in (61) at $T = 0$. We will refer to this as the “stated ranges” for η and T . For an integer $R \geq 0$, define

$$(74) \quad J_{T,R}(\eta) = \frac{1}{1-q^{-1}} \int_{F^\times} \int_{\varpi^{-R}\mathfrak{o}} \int_F \int_F \eta(a) |a|^{3/2} \psi(\varpi z) \Phi_{0,T}(g(a; x, y, z)) dx dy dz d^\times a.$$

Lemma 2.8. *For every fixed integer $R \geq 0$, the integral (74) is absolutely convergent in the stated ranges. For fixed η , the function $T \mapsto J_{T,R}(\eta)$ is holomorphic on $|T| < 1$, and, when $s_0 < \frac{1}{2}$,*

$$\lim_{T \uparrow 1} J_{T,R}(\eta) = J_{1,R}(\eta).$$

Moreover, for $|T| < q^{-3}$, the integral

$$(75) \quad \frac{1}{1-q^{-1}} \int_{F^\times} \int_F \int_F \int_F \eta(a) |a|^{3/2} \psi(\varpi z) \Phi_{0,T}(g(a; x, y, z)) dx dy dz d^\times a$$

is absolutely convergent, including when $s_0 = \frac{1}{2}$.

Proof. Since $g(a; x, y, z) = g(a; x, y, 0)g(1; 0, 0, z)$, bounded z changes Cartan exponents by bounded amounts. Put $k = v(a)$, $n = |k|$, and $(X, Y) = (a^2 x, ay)$ for $k \geq 0$, or $(X, Y) = (x, y)$ for $k < 0$. Set $U = \log_q \max(1, |X|)$, $V = \log_q \max(1, |Y|)$, $\mu = \max(U, V)$ and $\nu = \max(U, 2V)$. The entry and two-by-two-minor maxima of $g(a; x, y, 0)$ are $q^{n+\mu}$ and $q^{n+\nu}$. Thus (62), the (X, Y) -shell volume bound q^{U+V} , and $U + V - \mu - \nu \leq -\mu/2$ bound the absolute integral in (74), for $|T| \leq t \leq 1$ with $t > 0$, by

$$(76) \quad C_{R,t} \sum_{n \geq 0} (1+n)^D (tq^{s_0-1/2})^n,$$

where $D > 0$ is fixed and $C_{R,t} > 0$ is independent of T . This proves the first assertions by local uniform domination and dominated convergence.

For the last assertion, let $N \geq 0$ be an integer. The entry bounds on $\text{ht}(g(a; x, y, z)) \leq N$ give $|k| \leq N$, $|x| \leq q^{N+k}$, $|y| \leq q^{N+\min(k,0)}$, and $|z| \leq q^N$. Their volume, weighted by $|\eta(a)||a|^{3/2}$, is at most

$$(1-q^{-1})q^{3N} \sum_{k=-N}^N q^{-(s_0+1/2)k+\min(k,0)} \leq (1-q^{-1})(2N+1)q^{3N}.$$

Since $|\Phi_0| \leq 1$, we obtain absolute convergence of (75) for $|T| < q^{-3}$. $\square$

For integers $r \geq 0$, put

$$\mathcal{Z}_0 = \mathfrak{o}, \quad \mathcal{Z}_r = \varpi^{-r} \mathfrak{o}^\times \quad (r \geq 1),$$

and define

$$(77) \quad I_{r,T}(\eta) = \frac{1}{1-q^{-1}} \int_{F^\times} \int_{\mathcal{Z}_r} \int_F \int_F \eta(a) |a|^{3/2} \psi(\varpi z) \Phi_{0,T}(g(a; x, y, z)) dx dy dz d^\times a.$$

For $s_0 < \frac{1}{2}$, also put $I_r(\eta) := I_{r,1}(\eta)$. These integrals are absolutely convergent by Lemma 2.8, and

$$J_{T,R}(\eta) = \sum_{r=0}^R I_{r,T}(\eta).$$

For $z \in \mathfrak{o}$, right K -invariance gives

$$(78) \quad \begin{aligned} I_{0,T}(\eta) &= \frac{1}{1-q^{-1}} \int_{F^\times} \int_F \int_F \eta(a) |a|^{3/2} \Phi_{0,T}(g(a; x, y, 0)) dx dy d^\times a, \\ I_0(\eta) &= \frac{1}{1-q^{-1}} \int_{F^\times} \int_F \int_F \eta(a) |a|^{3/2} \Phi_0(g(a; x, y, 0)) dx dy d^\times a \quad (s_0 < \tfrac{1}{2}). \end{aligned}$$

Lemma 2.9. *In the stated ranges for η and T , we have*

$$(79) \quad I_{1,T}(\eta) = q \int_{F^\times} \int_F \int_F \eta(a) |a|^{3/2} \Phi_{0,T}(g(a; x, y, \varpi^{-1})) dx dy d^\times a,$$

$$(80) \quad I_{2,T}(\eta) = -\frac{q}{1-q^{-1}} \int_{F^\times} \int_F \int_F \eta(a) |a|^{3/2} \Phi_{0,T}(g(a; x, y, \varpi^{-2})) dx dy d^\times a,$$

and $I_{r,T}(\eta) = 0$ for $r \geq 3$. In particular, for $s_0 < \frac{1}{2}$, setting $T = 1$ gives the formulas for $I_1(\eta)$ and $I_2(\eta)$, and $I_r(\eta) = 0$ for $r \geq 3$.

Proof. Writing $z = \varpi^{-r} b$ with $b \in \mathfrak{o}^\times$, an easy calculation gives

$$(81) \quad \begin{aligned} (1-q^{-1})I_{r,T}(\eta) &= q^r \left(\int_{\mathfrak{o}^\times} \psi(\varpi^{1-r} b) d^\times b \right) \\ &\quad \times \int_{F^\times} \int_F \int_F \eta(a) |a|^{3/2} \Phi_{0,T}(g(a; x, y, \varpi^{-r})) dx dy d^\times a. \end{aligned}$$

The integral over $\mathfrak{o}^\times$ is $1-q^{-1}$ for $r = 1$, $-q^{-1}$ for $r = 2$, and zero for $r \geq 3$. $\square$

For every integer $R \geq 2$, Lemma 2.9 gives

$$J_{T,R}(\eta) = I_{0,T}(\eta) + I_{1,T}(\eta) + I_{2,T}(\eta).$$

Thus, for $0 \leq s_0 < \frac{1}{2}$, we may define

$$(82) \quad I(\eta) := \lim_{R \rightarrow \infty} J_{1,R}(\eta) = J_{1,2}(\eta) = I_0(\eta) + I_1(\eta) + I_2(\eta),$$

and, for $0 \leq s_0 \leq \frac{1}{2}$ and $|T| < 1$, we define

$$(83) \quad I_T(\eta) := \lim_{R \rightarrow \infty} J_{T,R}(\eta) = J_{T,2}(\eta) = I_{0,T}(\eta) + I_{1,T}(\eta) + I_{2,T}(\eta).$$

Lemma 2.8 shows that $T \mapsto I_T(\eta)$ is holomorphic on $|T| < 1$ and that

$$\lim_{T \uparrow 1} I_T(\eta) = I(\eta) \quad (0 \leq s_0 < \tfrac{1}{2}).$$

For $|T| < q^{-3}$, absolute convergence identifies $I_T(\eta)$ with (75). By contrast, $I(\eta)$ is defined by stabilization of the central cutoff; no absolute convergence of (75) at $T = 1$ is asserted.

Proposition 2.10. *Use the core integrals $I(\eta)$ and $I_T(\eta)$ defined in (82) and (83).*

i) Let $\sigma = \tilde{\pi}(\chi)$ be an unramified principal series representation, with χ unitary. For $0 < T < 1$,

$$(84) \quad I(\Phi_{0,T}^m, \Phi_1^J) = I(\Phi_{0,T}^m, \Phi_2^J) = \frac{I_T(\chi)}{q^2(q+1)}.$$

Moreover,

$$(85) \quad \alpha(\pi, \sigma; m) = \frac{2}{q+1} \lim_{T \uparrow 1} I_T(\chi) = \frac{2}{q+1} I(\chi).$$

ii) Let $\xi \in \mathfrak{o}^\times$, $\eta_{s_0} = (\cdot, \xi) | \cdot |^{s_0}$, $0 \leq s_0 \leq 1/2$, and $0 < T < 1$. Then

$$(86) \quad I(\Phi_{0,T}^m, \Phi_{1,s_0}^J) = I(\Phi_{0,T}^m, \Phi_{2,s_0}^J) = \frac{I_T(\eta_{s_0})}{q^2(q+1)}.$$

The cross terms defined in (64) and (65) satisfy

$$(87) \quad \mathcal{C}_{21,s_0}(\Phi_{0,T}^m) = \frac{q^{s_0+1/2}(\varpi, \xi)}{q+1} I(\Phi_{0,T}^m, \Phi_{1,s_0}^J),$$

$$(88) \quad \mathcal{C}_{12,s_0}(\Phi_{0,T}^m) = \frac{q^{-s_0+1/2}(\varpi, \xi)}{q+1} I(\Phi_{0,T}^m, \Phi_{1,s_0}^J).$$

Moreover, for the special representation $\sigma = \tilde{\sigma}_\xi$, with $\varepsilon_\sigma = (\varpi, \xi)$,

$$(89) \quad \alpha(\pi, \sigma; m) = \lim_{T \uparrow 1} \frac{1 - \varepsilon_\sigma}{q+1} I_T(\eta_{1/2}).$$

Proof. First fix $0 < T < q^{-6}$. Lemma 2.7 iii) justifies expanding relevant integration formulas in the pairings and applying Fubini; (75) is also absolutely convergent in this range. All calculations below are initially made in this range of T .

Below, we will use the sign symmetry

$$(90) \quad \Phi_{0,T}(g(a; -x, y, -z)) = \Phi_{0,T}(g(a; x, y, z)),$$

which follows by conjugation with $e_{-1} \in K$. In residue-class sums below we use fixed representatives in $\mathfrak{o}$, with 0 representing the zero class.

i) We use the subgroup $K_1^J(\mathfrak{p})$ defined in (39), under which $\Phi_{0,T}^m$ is left and right invariant. Starting from the integral (48) with Φ_0^m replaced by $\Phi_{0,T}^m$, and changing variables $g \mapsto \kappa^{-1} \begin{bmatrix} 1 & -u \\ & 1 \\ & & 1 & u \\ & & & 1 \end{bmatrix} g$, we obtain

$$(91) \quad I(\Phi_{0,T}^m, \Phi_i^J) = \frac{1}{\text{vol}(B_1(\mathfrak{o})g_i\Gamma_0(\mathfrak{p}))} \int_{B_1(\mathfrak{o})g_i\Gamma_0(\mathfrak{p})} \int_{\mathfrak{o}} \int_{J(F)} \Phi_{0,T}^m \left(\kappa^{-1} \begin{bmatrix} 1 & -u \\ & 1 \\ & & 1 & u \\ & & & 1 \end{bmatrix} g \right) f_i(g) dg du d\kappa,$$

where, as before, $g_1 = 1$ and $g_2 = w$ (see (15)). For $i = 1$, we have $B_1(\mathfrak{o})\Gamma_0(\mathfrak{p}) = \Gamma_0(\mathfrak{p})$. Since $\Phi_{0,T}^m$ is left invariant by $K_1^J(\mathfrak{p})$, and since $\Gamma_0(\mathfrak{p}) \subset K_1^J(\mathfrak{p})$, while

$$\begin{bmatrix} 1 & v \\ & 1 \\ & & 1 & -v \\ & & & 1 \end{bmatrix} \in K_1^J(\mathfrak{p}) \quad \text{for } v \in \mathfrak{p},$$

we get

$$(92) \quad I(\Phi_{0,T}^m, \Phi_1^J) = q^{-1} \sum_{r \in \mathfrak{o}/\mathfrak{p}} \int_{J(F)} \Phi_{0,T}^m \left(\begin{bmatrix} 1 & -r \\ & 1 \\ & & 1 & r \\ & & & 1 \end{bmatrix} g \right) f_1(g) dg.$$

Write the $J(F)$ integral as a nested $\mathrm{SL}_2(F)$ and $H(F)$ integral. Using (17), and the fact that f_1 is supported on $B^J(F)K_0^J(\mathfrak{p})$, we get

$$\begin{aligned} & \int_{J(F)} \Phi_{0,T}^m \left(\begin{bmatrix} 1 & -r & & \\ & 1 & & \\ & & 1 & r \\ & & & 1 \end{bmatrix} g \right) f_1(g) dg \\ &= \frac{\mathrm{vol}(\Gamma_0(\mathfrak{p}))}{1-q^{-1}} \int_{F^\times} \int_F \int_{H(F)} \Phi_{0,T}^m \left(\begin{bmatrix} 1 & -r & & \\ & 1 & & \\ & & 1 & r \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & a & & \\ & a^{-1} & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} h \right) \\ & \quad \times f_1 \left(\begin{bmatrix} 1 & a & & \\ & a^{-1} & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} h \right) dh dx d^\times a. \end{aligned}$$

Write

$$(93) \quad h = \begin{bmatrix} 1 & y & z \\ & 1 & y \\ & & 1 \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & u & & \\ & 1 & & \\ & & 1 & -u \\ & & & 1 \end{bmatrix}.$$

Using the defining transformation property of $\pi^J(\chi)$, we get

$$f_1 \left(\begin{bmatrix} 1 & a & & \\ & a^{-1} & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & y & z \\ & 1 & y \\ & & 1 \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & u & & \\ & 1 & & \\ & & 1 & -u \\ & & & 1 \end{bmatrix} \right) = \begin{cases} \chi(a)|a|^{3/2}\psi^{-1}(z), & u \in \mathfrak{o}, \\ 0, & u \notin \mathfrak{o}. \end{cases}$$

The u -integration now becomes, because of the right invariance of $\Phi_{0,T}^m$ by $K_1^J(\mathfrak{p})$, a sum over $u \in \mathfrak{o}/\mathfrak{p}$. We claim that only the class $u = 0$ contributes. For $t \in \mathfrak{p}^{-1}$, the element

$$(94) \quad \begin{bmatrix} 1 & & & \\ & 1 & -t & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

belongs to $K_1^J(\mathfrak{p})$. Inserting this element to the right of the argument of $\Phi_{0,T}^m$ and using the identity

$$\begin{bmatrix} 1 & y & z \\ & 1 & y \\ & & 1 \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & u & & \\ & 1 & & \\ & & 1 & -u \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & -t & \\ & & 1 & \\ & & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & -t & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & y-tu & z-tu^2 \\ & 1 & y-tu \\ & & 1 \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & u & & \\ & 1 & & \\ & & 1 & -u \\ & & & 1 \end{bmatrix},$$

we see that each u -summand is equal to itself multiplied by $\psi^{-1}(tu^2)$. Since ψ has conductor $\mathfrak{o}$, this happens if and only if $u \in \mathfrak{p}$. So we get,

$$\begin{aligned} & \int_{J(F)} \Phi_{0,T}^m \left(\begin{bmatrix} 1 & -r & & \\ & 1 & & \\ & & 1 & r \\ & & & 1 \end{bmatrix} g \right) f_1(g) dg \\ &= \frac{\mathrm{vol}(\Gamma_0(\mathfrak{p}))}{q(1-q^{-1})} \int_{F^\times} \int_F \int_F \int_F \chi(a)|a|^{3/2}\psi^{-1}(z) \\ & \quad \times \Phi_{0,T}^m \left(\begin{bmatrix} 1 & -r & & \\ & 1 & & \\ & & 1 & r \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & a & & \\ & a^{-1} & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & y & z \\ & 1 & y \\ & & 1 \\ & & & 1 \end{bmatrix} \right) dx dy dz d^\times a. \end{aligned}$$

Next, for any $t \in \mathfrak{p}^{-1}$, we insert the inverse of the element (94) to the left of the argument of $\Phi_{0,T}^m$, using the left $K_1^J(\mathfrak{p})$ -invariance of $\Phi_{0,T}^m$. The identity

$$\begin{bmatrix} 1 & & & \\ & 1 & t & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & -r & & \\ & 1 & & \\ & & 1 & r \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & a & & \\ & a^{-1} & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & y & z \\ & 1 & y \\ & & 1 \\ & & & 1 \end{bmatrix} = \begin{bmatrix} 1 & -r & & \\ & 1 & & \\ & & 1 & r \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & a & & \\ & a^{-1} & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & y+a^{-1}rt & z+r^2t \\ & 1 & x+a^{-2}t & y+a^{-1}rt \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

shows that the integral is equal to itself multiplied by $\psi(r^2t)$. Therefore only the class $r = 0$ contributes to the sum over $\mathfrak{o}/\mathfrak{p}$, showing that

$$(95) \quad \begin{aligned} I(\Phi_{0,T}^m, \Phi_1^J) &= \frac{1}{q^2(q+1)(1-q^{-1})} \int_{F^\times} \int_F \int_F \int_F \chi(a)|a|^{3/2}\psi^{-1}(z) \\ & \quad \times \Phi_{0,T}^m \left(\begin{bmatrix} 1 & a & & \\ & a^{-1} & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & y & z \\ & 1 & y \\ & & 1 \\ & & & 1 \end{bmatrix} \right) dx dy dz d^\times a. \end{aligned}$$

Now, using $\Phi_{0,T}^m(g) = \Phi_{0,T}(e_{\varpi^{-1}}ge_{\varpi})$ (see (34)), making a further change of variables $z \mapsto -\varpi z$, $x \mapsto -\varpi^{-1}x$, and applying (75), we obtain the expression for $I(\Phi_{0,T}^m, \Phi_1^J)$ in (84).

For $i = 2$, we use the decomposition

$$B_1(\mathfrak{o}) \begin{bmatrix} 1 & & \\ & -1 & 1 \\ & & 1 \end{bmatrix} \Gamma_0(\mathfrak{p}) = \bigsqcup_{s \in \mathfrak{o}/\mathfrak{p}} \begin{bmatrix} 1 & & s \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & -1 & 1 \\ & & 1 \end{bmatrix} \Gamma_0(\mathfrak{p}).$$

Thus, writing

$$\kappa = \begin{bmatrix} 1 & & s \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & -1 & 1 \\ & & 1 \end{bmatrix} \gamma, \quad \gamma \in \Gamma_0(\mathfrak{p}),$$

using the left $K_1^J(\mathfrak{p})$ -invariance of $\Phi_{0,T}^m$ and replacing the u -integration by a sum over $r \in \mathfrak{o}/\mathfrak{p}$, we obtain from (91)

$$(96) \quad I(\Phi_{0,T}^m, \Phi_2^J) = q^{-2} \sum_{r,s \in \mathfrak{o}/\mathfrak{p}} \int_{J(F)} \Phi_{0,T}^m \left(\begin{bmatrix} 1 & & \\ & 1 & -1 \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & s \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & -r & \\ & 1 & r \\ & & 1 \end{bmatrix} g \right) f_2(g) dg.$$

As before, writing the $J(F)$ -integral as a nested $\mathrm{SL}_2(F)$ and $H(F)$ integral and using (17), we get

$$\begin{aligned} & \int_{J(F)} \Phi_{0,T}^m \left(\begin{bmatrix} 1 & & \\ & 1 & -1 \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & s \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & -r & \\ & 1 & r \\ & & 1 \end{bmatrix} g \right) f_2(g) dg = \frac{q}{(q+1)(1-q^{-1})} \\ & \times \int_{F^\times} \int_F \int_{H(F)} \Phi_{0,T}^m \left(\begin{bmatrix} 1 & & \\ & 1 & -1 \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & s \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & -r & \\ & 1 & r \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & a & \\ & a^{-1} & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & x \\ & 1 & \\ & & 1 \end{bmatrix} h \begin{bmatrix} 1 & & \\ & -1 & 1 \\ & & 1 \end{bmatrix} \right) \\ & \times f_2 \left(\begin{bmatrix} 1 & a & \\ & a^{-1} & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & x \\ & 1 & \\ & & 1 \end{bmatrix} h \begin{bmatrix} 1 & & \\ & -1 & 1 \\ & & 1 \end{bmatrix} \right) dh dx d^\times a. \end{aligned}$$

Above, we have used $\mathrm{vol}(B_1(\mathfrak{o})w\Gamma_0(\mathfrak{p})) = \frac{q}{q+1}$. As before we write h in the form (93). Observe that

$$f_2 \left(\begin{bmatrix} 1 & a & \\ & a^{-1} & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & x \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & y & z \\ & 1 & y \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & u & \\ & 1 & -u \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & -1 & 1 \\ & & 1 \end{bmatrix} \right) = \begin{cases} \chi(a)|a|^{3/2}\psi^{-1}(z), & u \in \mathfrak{o}, \\ 0, & u \notin \mathfrak{o}. \end{cases}$$

The u -dependence disappears from the argument of $\Phi_{0,T}^m$ by right $K_1^J(\mathfrak{p})$ -invariance, and the u -integration contributes $\mathrm{vol}(\mathfrak{o}) = 1$. Combining this with (96), we obtain

$$(97) \quad \begin{aligned} I(\Phi_{0,T}^m, \Phi_2^J) &= \frac{1}{q(q+1)(1-q^{-1})} \sum_{r,s \in \mathfrak{o}/\mathfrak{p}} \int_{F^\times} \int_F \int_F \int_F \chi(a)|a|^{3/2}\psi^{-1}(z) \\ &\times \Phi_{0,T}^m \left(\begin{bmatrix} 1 & & \\ & 1 & -1 \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & s \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & -r & \\ & 1 & r \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & a & \\ & a^{-1} & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & y & z \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & -1 & 1 \\ & & 1 \end{bmatrix} \right) \\ &\times dx dy dz d^\times a. \end{aligned}$$

Inserting the element $\begin{bmatrix} 1 & & r \\ & 1 & \\ & & 1 \end{bmatrix} \in K_1^J(\mathfrak{p})$ on the left and performing the translations $x \mapsto x - \frac{s}{a^2}$,

$y \mapsto y - \frac{rs}{a}$, $z \mapsto z - r^2s$ shows that all summands are equal. Hence

$$(98) \quad I(\Phi_{0,T}^m, \Phi_2^J) = \frac{q}{(q+1)(1-q^{-1})} \int_{F^\times} \int_F \int_F \int_F \chi(a)|a|^{3/2}\psi^{-1}(z) \Phi_{0,T}^m \left(\begin{bmatrix} 1 & -y & 0 & z \\ & a^{-1} & 0 & 0 \\ & -ax & a & ay \\ & & & 1 \end{bmatrix} \right) dx dy dz d^\times a.$$

Now, using $\Phi_{0,T}^m(g) = \Phi_{0,T}(w^{-1}e_{\varpi^{-1}}ge_{\varpi}w)$ (see (15) and (34)), making a further change of variables $z \mapsto -\varpi z$, $x \mapsto -\varpi x$, $y \mapsto \varpi y$, and applying (90), we obtain the expression given in (84) in the initial range of T .

ii) For the special representation case, fix $0 \leq s_0 \leq \frac{1}{2}$. The preceding computations of the diagonal terms apply verbatim after replacing χ by η_{s_0} and f_i by f_{i,s_0} . They give the asserted formula for $I(\Phi_{0,T}^m, \Phi_{1,s_0}^J) = I(\Phi_{0,T}^m, \Phi_{2,s_0}^J)$.

We now compute $\mathcal{C}_{21,s_0}(\Phi_{0,T}^m)$. As in the computation of $I(\Phi_{0,T}^m, \Phi_{1,s_0}^J)$, starting from (66), we obtain

$$(99) \quad \mathcal{C}_{21,s_0}(\Phi_{0,T}^m) = \frac{1}{q(q+1)} \sum_{r \in \mathfrak{o}/\mathfrak{p}} \int_{J(F)} \Phi_{0,T}^m \left(\begin{bmatrix} 1 & -r & & \\ & 1 & & \\ & & 1 & r \\ & & & 1 \end{bmatrix} g \right) f_{2,s_0}(g) dg.$$

Writing the $J(F)$ -integral as a nested $\mathrm{SL}_2(F)$ - and $H(F)$ -integral, using the formula (17), and using the support of f_{2,s_0} , we obtain

$$(100) \quad \begin{aligned} \mathcal{C}_{21,s_0}(\Phi_{0,T}^m) &= \frac{1}{(q+1)^2(1-q^{-1})} \sum_{r \in \mathfrak{o}/\mathfrak{p}} \int_{F^\times} \int_F \int_F \int_F \eta_{s_0}(a) |a|^{3/2} \psi^{-1}(z) \\ &\quad \times \Phi_{0,T}^m \left(\begin{bmatrix} 1 & -r & & \\ & 1 & & \\ & & 1 & r \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & a & & \\ & a^{-1} & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & y & z \\ & 1 & y \\ & & 1 \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & -1 & 1 & \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) dx dy dz d^\times a. \end{aligned}$$

The same argument as in the derivation of (84) shows that only the class $r = 0$ contributes. Hence

$$(101) \quad \begin{aligned} \mathcal{C}_{21,s_0}(\Phi_{0,T}^m) &= \frac{1}{(q+1)^2(1-q^{-1})} \int_{F^\times} \int_F \int_F \int_F \eta_{s_0}(a) |a|^{3/2} \psi^{-1}(z) \\ &\quad \times \Phi_{0,T}^m \left(\begin{bmatrix} 1 & a & & \\ & a^{-1} & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & y & z \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & -1 & 1 & \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) dx dy dz d^\times a. \end{aligned}$$

Passing from $\Phi_{0,T}^m$ to $\Phi_{0,T}$, using the right K -invariance of $\Phi_{0,T}$, and making the same harmless sign change $x \mapsto -x$, $z \mapsto -z$ as in the derivation of (84), produces the factor $q^{-3/2} \eta_{s_0}(\varpi)^{-1}$. Comparing with (86) then gives (87).

We now compute the second cross term. Starting from (67) and making the same change of variables gives

$$(102) \quad \mathcal{C}_{12,s_0}(\Phi_{0,T}^m) = \int_{B_1(\mathfrak{o})w\Gamma_0(\mathfrak{p})} \int_{\mathfrak{o}} \int_{J(F)} \Phi_{0,T}^m \left(\kappa^{-1} \begin{bmatrix} 1 & -u & & \\ & 1 & & \\ & & 1 & u \\ & & & 1 \end{bmatrix} g \right) f_{1,s_0}(g) dg du d\kappa.$$

By the same sequence of steps used in the computation of $I(\Phi_{0,T}^m, \Phi_{2,s_0}^J)$, we reduce to

$$\begin{aligned} \mathcal{C}_{12,s_0}(\Phi_{0,T}^m) &= \frac{1}{(q+1)^2(1-q^{-1})} \int_{F^\times} \int_F \int_F \int_F \eta_{s_0}(a) |a|^{3/2} \psi^{-1}(z) \\ &\quad \times \Phi_{0,T}^m \left(\begin{bmatrix} 1 & & & \\ & 1 & -1 & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & a & & \\ & a^{-1} & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & y & z \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) dx dy dz d^\times a. \end{aligned}$$

Finally,

$$e_{\varpi^{-1}} \begin{bmatrix} 1 & & & \\ & 1 & -1 & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & a & & \\ & a^{-1} & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & y & z \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix} e_{\varpi} = \begin{bmatrix} 1 & & & \\ & 1 & -1 & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & a\varpi^{-1} & & \\ & a^{-1}\varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & y & \varpi^{-1}z \\ & 1 & \varpi x & y \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

The first factor lies in K , so it may be removed from the argument of $\Phi_{0,T}$. With the changes of variables

$$a \mapsto a\varpi, \quad x \mapsto \varpi^{-1}x, \quad z \mapsto \varpi z,$$

followed by $x \mapsto -x$, $z \mapsto -z$ and (90), we obtain

$$\begin{aligned} \mathcal{C}_{12,s_0}(\Phi_{0,T}^m) &= \frac{q^{-3/2} \eta_{s_0}(\varpi)}{(q+1)^2(1-q^{-1})} \int_{F^\times} \int_F \int_F \int_F \eta_{s_0}(a) |a|^{3/2} \psi(\varpi z) \\ &\quad \times \Phi_{0,T} \left(\begin{bmatrix} 1 & a & & \\ & a^{-1} & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & y & z \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) dx dy dz d^\times a. \end{aligned}$$

Comparing with (86) gives

$$\mathcal{C}_{12,s_0}(\Phi_{0,T}^m) = \frac{q^{1/2}\eta_{s_0}(\varpi)}{q+1} I(\Phi_{0,T}^m, \Phi_{1,s_0}^J) = \frac{q^{-s_0+1/2}(\varpi, \xi)}{q+1} I(\Phi_{0,T}^m, \Phi_{1,s_0}^J),$$

which proves (88).

The identities have now been proved for $0 < T < q^{-6}$, including at $s_0 = \frac{1}{2}$. The period integrals on the left of (84) are holomorphic on this disk by Lemma 2.7 ii), and the diagonal and cross integrals in part ii) are holomorphic there by part i) of that lemma. The functions $I_T(\chi)$ and $I_T(\eta_{s_0})$ are holomorphic there by Lemma 2.8 and (83). The identities (84), (86), (87), (88) therefore extend to $0 < T < 1$.

In the principal-series case, Lemma 2.7 ii), (47), (84) and $I_T(\chi) \rightarrow I(\chi)$ give (85). For $s_0 = \frac{1}{2}$, substitution of (86), (87), (88) into (70) gives the identity

$$q^2 I(\Phi_{0,T}^m, \Phi_{\text{sp}}^J) = \frac{1 - \varepsilon_\sigma}{q+1} I_T(\eta_{1/2}).$$

Taking $T \uparrow 1$ above, using Lemma 2.7 ii) and (60), proves (89). $\square$

2.7. Preparation for calculation of the core integral. Proposition 2.10 reduces the local periods to core integrals. In the unitary principal-series case, it gives

$$(103) \quad \alpha(\pi, \sigma; m) = \frac{2}{q+1} (I_0(\chi) + I_1(\chi) + I_2(\chi)).$$

For the special representation $\sigma \simeq \tilde{\sigma}_\xi$, it gives

$$(104) \quad \alpha(\pi, \sigma; m) = \lim_{T \uparrow 1} \frac{1 - \varepsilon_\sigma}{q+1} (I_{0,T}(\eta_{1/2}) + I_{1,T}(\eta_{1/2}) + I_{2,T}(\eta_{1/2})).$$

The next three subsections evaluate $I_{r,T}(\eta)$ for $r = 0, 1, 2$ throughout the ranges used in Lemma 2.8. More precisely, throughout these calculations we take

$$\eta = \eta_0 |\cdot|^{s_0}, \quad \eta_0 \text{ unitary and unramified}, \quad 0 \leq s_0 \leq \frac{1}{2},$$

and allow $|T| < 1$, as well as $T = 1$ when $s_0 < \frac{1}{2}$. The result in the principal-series is obtained by taking $\eta = \chi$ and $T = 1$, whereas the special case uses $\eta = \eta_{1/2}$ with $0 < T < 1$ before taking the limit in (104).

The strategy in each case is similar to the one employed in [11]: Use Proposition 3.2 of [21] and the Macdonald formula to explicitly evaluate the matrix coefficient, and then integrate the resulting quantity by reducing it to a sum.

For integers ℓ, m , put

$$(105) \quad h(\ell, m) = \begin{bmatrix} \varpi^{\ell+m} & & \\ & \varpi^{\ell+2m} & \\ & & 1 \\ & & & \varpi^m \end{bmatrix}.$$

Here and in the formulas and tables below, m is an integer index, not the fixed Fourier–Jacobi index in $\alpha(\pi, \sigma; m)$. Recall the Satake parameters α, β, γ of π , and put

$$(106) \quad \delta = \eta(\varpi), \quad |\delta| = q^{-s_0}.$$

We also put

$$(107) \quad D_q = 1 + 2q^{-1} + 2q^{-2} + 2q^{-3} + q^{-4}.$$

We first carry out the calculations for *regular* unitary Satake parameters α, β , meaning that none of the denominators of the A_i in (109) vanishes. The extension to all unitary parameters is explained later.

The following is the required special case of the Macdonald formula.

Proposition 2.11. *For all non-negative integers ℓ, m , we have*

$$(108) \quad \Phi_0(h(\ell, m)) = \frac{q^{-(4m+3\ell)/2} \gamma^\ell}{D_q} \sum_{i=1}^8 A_i B_i(\ell, m),$$

where the quantities A_i, B_i are given as follows:

i	A_i	$B_i(\ell, m)$	L_i	M_i
1	$\frac{1-q^{-1}\alpha^{-1}\beta}{1-\alpha^{-1}\beta} \frac{1-q^{-1}\beta^{-1}}{1-\beta^{-1}} \frac{1-q^{-1}\alpha^{-1}\beta^{-1}}{1-\alpha^{-1}\beta^{-1}} \frac{1-q^{-1}\alpha^{-1}}{1-\alpha^{-1}}$	$\alpha^{m+\ell}\beta^\ell$	$\alpha\beta$	α
2	$\frac{1-q^{-1}\alpha\beta^{-1}}{1-\alpha\beta^{-1}} \frac{1-q^{-1}\alpha^{-1}}{1-\alpha^{-1}} \frac{1-q^{-1}\alpha^{-1}\beta^{-1}}{1-\alpha^{-1}\beta^{-1}} \frac{1-q^{-1}\beta^{-1}}{1-\beta^{-1}}$	$\alpha^\ell\beta^{m+\ell}$	$\alpha\beta$	β
3	$\frac{1-q^{-1}\alpha^{-1}\beta^{-1}}{1-\alpha^{-1}\beta^{-1}} \frac{1-q^{-1}\beta}{1-\beta} \frac{1-q^{-1}\alpha^{-1}\beta}{1-\alpha^{-1}\beta} \frac{1-q^{-1}\alpha^{-1}}{1-\alpha^{-1}}$	$\alpha^{m+\ell}$	α	α
4	$\frac{1-q^{-1}\alpha\beta}{1-\alpha\beta} \frac{1-q^{-1}\alpha^{-1}}{1-\alpha^{-1}} \frac{1-q^{-1}\alpha^{-1}\beta}{1-\alpha^{-1}\beta} \frac{1-q^{-1}\beta}{1-\beta}$	$\alpha^\ell\beta^{-m}$	α	β^{-1}
5	$\frac{1-q^{-1}\alpha^{-1}\beta^{-1}}{1-\alpha^{-1}\beta^{-1}} \frac{1-q^{-1}\alpha}{1-\alpha} \frac{1-q^{-1}\alpha\beta^{-1}}{1-\alpha\beta^{-1}} \frac{1-q^{-1}\beta^{-1}}{1-\beta^{-1}}$	$\beta^{m+\ell}$	β	β
6	$\frac{1-q^{-1}\alpha\beta}{1-\alpha\beta} \frac{1-q^{-1}\beta^{-1}}{1-\beta^{-1}} \frac{1-q^{-1}\alpha\beta^{-1}}{1-\alpha\beta^{-1}} \frac{1-q^{-1}\alpha}{1-\alpha}$	$\alpha^{-m}\beta^\ell$	β	α^{-1}
7	$\frac{1-q^{-1}\alpha^{-1}\beta}{1-\alpha^{-1}\beta} \frac{1-q^{-1}\alpha}{1-\alpha} \frac{1-q^{-1}\alpha\beta}{1-\alpha\beta} \frac{1-q^{-1}\beta}{1-\beta}$	β^{-m}	1	β^{-1}
8	$\frac{1-q^{-1}\alpha\beta^{-1}}{1-\alpha\beta^{-1}} \frac{1-q^{-1}\beta}{1-\beta} \frac{1-q^{-1}\alpha\beta}{1-\alpha\beta} \frac{1-q^{-1}\alpha}{1-\alpha}$	α^{-m}	1	α^{-1}

(Note that the above is slightly different from the statement in [11]. The simplification $\alpha\beta\gamma^2 = 1$ has been applied, and γ^ℓ has been factored out.)

The significance of the quantities L_i, M_i in table (109) is that

$$(110) \quad B_i(\ell, m) = L_i^\ell M_i^m.$$

We use $v(0) = +\infty$ in the valuation formulas below. Recall that

$$g(a; x, y, z) = \begin{bmatrix} 1 & a & & \\ & a^{-1} & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & y & z \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

The following is a special case of Proposition 3.2 of [21].

Lemma 2.12. *Let $a \in F^\times$ and $x, y, z \in F$. Let*

$$(111) \quad g = g(a; x, y, z).$$

Define $\ell_1, \ell_2 \in \mathbb{Z}$ by

$$(112) \quad \ell_1 = \min\{0, v(a), -v(a), v(ax), v(y), v(ay), v(z)\},$$

$$(113) \quad \ell_1 + \ell_2 = \min\{0, v(a), -v(a), v(ax), v(y), v(ay), v(z) + v(a), v(z) - v(a), v(a) + v(y^2 - xz)\}.$$

Then $\ell_1 \leq \ell_2 \leq 0$ and

$$(114) \quad g \in K \begin{bmatrix} \varpi^{\ell_1} & & & \\ & \varpi^{\ell_2} & & \\ & & \varpi^{-\ell_2} & \\ & & & \varpi^{-\ell_1} \end{bmatrix} K.$$

Remark 2.13. *The inequality $\ell_1 \leq \ell_2$ is not stated explicitly in Proposition 3.2 of [21]. In fact, this proposition is formulated in such a way that $\ell_1 \leq \ell_2$ is a hypothesis. While not obvious, it can be checked directly that the definitions (112), (113) imply $\ell_1 \leq \ell_2$.*

We rewrite the double coset in (114) as

$$(115) \quad K \begin{bmatrix} \varpi^{-\ell_2} & & & \\ & \varpi^{-\ell_1} & & \\ & & \varpi^{\ell_1} & \\ & & & \varpi^{\ell_2} \end{bmatrix} K = \varpi^{\ell_1} K \begin{bmatrix} \varpi^{-\ell_1-\ell_2} & & & \\ & \varpi^{-2\ell_1} & & \\ & & 1 & \\ & & & \varpi^{\ell_2-\ell_1} \end{bmatrix} K$$

Hence, instead of (114), we can say that

$$(116) \quad g \in Z(F)Kh(\ell, m)K \quad \text{with } \ell = -2\ell_2, m = \ell_2 - \ell_1.$$

Above, $Z(F)$ denotes the center of $\mathrm{GSp}_4(F)$.

The height and Φ_0 are invariant under multiplication by $Z(F)$ and under left and right multiplication by K . In the setup of (116), we have

$$(117) \quad \mathrm{ht}(g) = m + \ell/2 = -\ell_1.$$

For $i = 1, \dots, 8$, define

$$(118) \quad U_i(T) := T\gamma^2 L_i^2, \quad V_i(T) := TM_i.$$

In the formulas below, U_i and V_i denote $U_i(T)$ and $V_i(T)$, respectively. Equations (108), (110) and (117) give

$$(119) \quad \Phi_{0,T}(g) = \frac{q^{-2m-3\ell/2}}{D_q} \sum_{i=1}^8 A_i U_i^{\ell/2} V_i^m \quad \text{for } g \in \mathrm{Sp}_4(F) \cap Z(F)Kh(\ell, m)K.$$

The geometric sums below have the following denominator factors:

$$(120) \quad \begin{aligned} &1 - \delta V_i q^{-1/2}, \quad 1 - \delta^{-1} V_i q^{-1/2}, \quad 1 - q^{-1} V_i^2, \\ &1 - q^{-1} U_i, \quad 1 - q^{-2} U_i. \end{aligned}$$

The corresponding ratios have absolute values

$$|T|q^{-s_0-1/2}, \quad |T|q^{s_0-1/2}, \quad |T|^2 q^{-1}, \quad |T|q^{-1}, \quad |T|q^{-2},$$

respectively. All are strictly less than one in the stated ranges. Thus these sums converge absolutely, including at $s_0 = \frac{1}{2}$ when $|T| < 1$, and locally uniformly in T on that disk.

2.8. Calculation of $I_{0,T}(\eta)$. We evaluate

$$(121) \quad I_{0,T}(\eta) = \frac{1}{1 - q^{-1}} \int_{F^\times} \int_F \int_F \eta(a) |a|^{3/2} \Phi_{0,T}(g(a; x, y, 0)) dx dy d^\times a,$$

as in (78). We use the following special case of Lemma 2.12.

Lemma 2.14. *Let $a \in F^\times$ and $x, y \in F$. Let*

$$(122) \quad g = g(a; x, y, 0).$$

Define $\ell_1, \ell_2 \in \mathbb{Z}$ by

$$(123) \quad \ell_1 = \min\{0, v(a), -v(a), v(ax), v(y), v(ay)\},$$

$$(124) \quad \ell_1 + \ell_2 = \min\{0, v(a), -v(a), v(ax), v(y), v(ay), v(a) + v(y^2)\}.$$

Then $\ell_1 \leq \ell_2 \leq 0$ and

$$(125) \quad g \in Z(F)Kh(\ell, m)K \quad \text{with } \ell = -2\ell_2, m = \ell_2 - \ell_1.$$

We divide the integration domain into ten regions. The last two columns give the indices ℓ and m in each region.

					ℓ	m
	1	$v(a) \geq 0$	$v(x) \geq -2v(a)$	$v(y) \geq -v(a)$	0	$v(a)$
	2			$v(y) < -v(a)$	$-2v(a) - 2v(y)$	$v(a)$
	3	$v(x) < -2v(a)$	$v(y) \geq v(a) + v(x)$	$2v(y) \geq v(x)$	0	$-v(a) - v(x)$
	4			$2v(y) < v(x)$	$2v(x) - 4v(y)$	$2v(y) - 2v(x) - v(a)$
(126)	5		$v(y) < v(a) + v(x)$		$-2v(a) - 2v(y)$	$v(a)$
	6	$v(a) < 0$	$v(x) \geq 0$	$v(y) \geq 0$	0	$-v(a)$
	7			$v(y) < 0$	$-2v(y)$	$-v(a)$
	8	$v(x) < 0$	$v(y) \geq v(x)$	$2v(y) \geq v(x)$	0	$-v(a) - v(x)$
	9			$2v(y) < v(x)$	$2v(x) - 4v(y)$	$2v(y) - 2v(x) - v(a)$
	10		$v(y) < v(x)$		$-2v(y)$	$-v(a)$

Now we calculate all ten pieces. We denote by $I_{0,j,T}(\eta)$ the piece of $I_{0,T}(\eta)$ corresponding to the j -th case. The computations are routine. We show the details only for Cases 1 and 2; the remaining cases are similar.

Case 1: For $k = v(a) \geq 0$, the x - and y -integrations have volumes q^{2k} and q^k , respectively, and the height is k . Hence

$$\begin{aligned}
 I_{0,1,T}(\eta) &= \frac{1}{1 - q^{-1}} \int_{v(a) \geq 0} \eta(a) |a|^{-3/2} T^{v(a)} \Phi_0(h(0, v(a))) d^\times a \\
 &= \sum_{k=0}^{\infty} \delta^k q^{3k/2} T^k \Phi_0(h(0, k)) \\
 &= \frac{1}{D_q} \sum_{i=1}^8 A_i \sum_{k=0}^{\infty} (\delta V_i q^{-1/2})^k \\
 (127) \quad &= \frac{1}{D_q} \sum_{i=1}^8 A_i \frac{1}{(1 - \delta V_i q^{-1/2})}.
 \end{aligned}$$

Case 2: Write $k = v(a) \geq 0$ and $v(y) = -k - j$ with $j \geq 1$. The x - and y -integrations have volume q^{2k} and $(1 - q^{-1})q^{k+j}$, and the Cartan indices are $(\ell, m) = (2j, k)$. This gives

$$\begin{aligned}
 I_{0,2,T}(\eta) &= (1 - q^{-1}) \sum_{k=0}^{\infty} \sum_{j=1}^{\infty} \delta^k q^{3k/2+j} T^{k+j} \Phi_0(h(2j, k)) \\
 &= \frac{1 - q^{-1}}{D_q} \sum_{i=1}^8 A_i \sum_{k=0}^{\infty} (\delta V_i q^{-1/2})^k \sum_{j=1}^{\infty} (q^{-2} U_i)^j \\
 (128) \quad &= \frac{(1 - q^{-1})q^{-2}}{D_q} \sum_{i=1}^8 A_i \frac{U_i}{(1 - \delta V_i q^{-1/2})(1 - q^{-2} U_i)}.
 \end{aligned}$$

The other eight cases are evaluated similarly.

Case 3:

$$(129) \quad I_{0,3,T}(\eta) = \frac{(1-q^{-1})q^{-1}}{D_q} \sum_{i=1}^8 A_i \frac{V_i(V_i+1)}{(1-\delta V_i q^{-1/2})(1-q^{-1}V_i^2)}.$$

Case 4:

$$(130) \quad I_{0,4,T}(\eta) = \frac{(1-q^{-1})^2 q^{-1}}{D_q} \sum_{i=1}^8 A_i \frac{U_i}{(1-\delta V_i q^{-1/2})(1-q^{-1}U_i)(1-q^{-1}V_i^2)}.$$

Case 5:

$$(131) \quad I_{0,5,T}(\eta) = \frac{(1-q^{-1})^2 q^{-3}}{D_q} \sum_{i=1}^8 A_i \frac{U_i^2}{(1-\delta V_i q^{-1/2})(1-q^{-1}U_i)(1-q^{-2}U_i)}.$$

Case 6:

$$(132) \quad I_{0,6,T}(\eta) = \frac{q^{-1/2}\delta^{-1}}{D_q} \sum_{i=1}^8 A_i \frac{V_i}{(1-\delta^{-1}V_i q^{-1/2})}.$$

Case 7:

$$(133) \quad I_{0,7,T}(\eta) = \frac{(1-q^{-1})q^{-5/2}\delta^{-1}}{D_q} \sum_{i=1}^8 A_i \frac{U_i V_i}{(1-\delta^{-1}V_i q^{-1/2})(1-q^{-2}U_i)}.$$

Case 8:

$$(134) \quad I_{0,8,T}(\eta) = \frac{(1-q^{-1})q^{-3/2}\delta^{-1}}{D_q} \sum_{i=1}^8 A_i \frac{V_i^2(V_i+1)}{(1-\delta^{-1}V_i q^{-1/2})(1-q^{-1}V_i^2)}.$$

Case 9:

$$(135) \quad I_{0,9,T}(\eta) = \frac{(1-q^{-1})^2 q^{-3/2}\delta^{-1}}{D_q} \sum_{i=1}^8 A_i \frac{U_i V_i}{(1-\delta^{-1}V_i q^{-1/2})(1-q^{-1}U_i)(1-q^{-1}V_i^2)}.$$

Case 10:

$$(136) \quad I_{0,10,T}(\eta) = \frac{(1-q^{-1})^2 q^{-7/2}\delta^{-1}}{D_q} \sum_{i=1}^8 A_i \frac{U_i^2 V_i}{(1-\delta^{-1}V_i q^{-1/2})(1-q^{-1}U_i)(1-q^{-2}U_i)}.$$

Adding the ten expressions gives the formula:

$$(137) \quad I_{0,T}(\eta) = \frac{1}{D_q} \sum_{i=1}^8 A_i \frac{(1-q^{-2}U_i)(1-q^{-2}V_i^2) + (1-q^{-1})q^{-1}V_i(1-q^{-1}U_i)}{(1-q^{-1}U_i)(1-\delta V_i q^{-1/2})(1-\delta^{-1}V_i q^{-1/2})}.$$

This identity holds throughout the stated ranges; setting $T = 1$ gives $I_0(\eta)$ when $s_0 < \frac{1}{2}$.

2.9. **Calculation of $I_{1,T}(\eta)$.** By (79), the integral to be evaluated is

$$(138) \quad I_{1,T}(\eta) = q \int_{F^\times} \int_F \int_F \eta(a) |a|^{3/2} \Phi_{0,T}(g(a; x, y, \varpi^{-1})) dx dy d^\times a.$$

We use the following special case of Lemma 2.12.

Lemma 2.15. *Let $a \in F^\times$ and $x, y \in F$. Let*

$$(139) \quad g = g(a; x, y, \varpi^{-1}).$$

Define $\ell_1, \ell_2 \in \mathbb{Z}$ by

$$(140) \quad \ell_1 = \min\{-1, v(a), -v(a), v(ax), v(y), v(ay)\},$$

$$(141) \quad \ell_1 + \ell_2 = \min\{0, v(a) - 1, -v(a) - 1, v(ax), v(y), v(ay), v(a) + v(y^2 - x\varpi^{-1})\}.$$

Then $\ell_1 \leq \ell_2 \leq 0$ and

$$(142) \quad g \in Z(F)Kh(\ell, m)K \quad \text{with } \ell = -2\ell_2, m = \ell_2 - \ell_1.$$

We divide the integration domain into fifteen regions. The last two columns give the indices ℓ and m in each region.

				ℓ	m
1	$v(a) \geq 1$	$v(x) \geq -2v(a)$	$v(y) \geq -v(a)$	2	$v(a) - 1$
2			$v(y) < -v(a)$	$-2v(a) - 2v(y)$	$v(a)$
3a		$v(x) < -2v(a)$	$v(y) \geq v(a) + v(x)$	2	$-v(a) - v(x) - 1$
3b			$2v(y) = v(x) - 1$	$\frac{y^2\varpi}{x} - 1 \in \mathfrak{o}^\times$	$-v(a) - v(x) - 1$
3c				$\frac{y^2\varpi}{x} - 1 \in \mathfrak{p}$	$-v(a) - v(x)$
4			$2v(y) \leq v(x) - 2$	$2v(x) - 4v(y)$	$2v(y) - 2v(x) - v(a)$
5			$v(y) < v(a) + v(x)$	$-2v(a) - 2v(y)$	$v(a)$
6a	$v(a) = 0$	$v(x) \geq 0$	$v(y) \geq 0$	0	1
6b	$v(a) \leq -1$	$v(x) \geq 0$	$v(y) \geq 0$	2	$-v(a) - 1$
7	$v(a) \leq 0$	$v(x) \geq 0$	$v(y) < 0$	$-2v(y)$	$-v(a)$
8a		$v(x) < 0$	$v(y) \geq v(x)$	2	$-v(a) - v(x) - 1$
8b			$2v(y) = v(x) - 1$	$\frac{y^2\varpi}{x} - 1 \in \mathfrak{o}^\times$	$-v(a) - v(x) - 1$
8c				$\frac{y^2\varpi}{x} - 1 \in \mathfrak{p}$	$-v(a) - v(x)$
9			$2v(y) \leq v(x) - 2$	$2v(x) - 4v(y)$	$2v(y) - 2v(x) - v(a)$
10			$v(y) < v(x)$	$-2v(y)$	$-v(a)$

Now we calculate all fifteen pieces. We denote by $I_{1,j,T}(\eta)$ the piece of $I_{1,T}(\eta)$ corresponding to the j -th case. We show the details only for Cases 1; the remaining cases are similar.

Case 1: Here $k = v(a) \geq 1$, $(\ell, m) = (2, k - 1)$ and the height is k . The x - and y -integrations have total volume q^{3k} , so

$$I_{1,1,T}(\eta) = q(1 - q^{-1}) \sum_{k=1}^{\infty} \delta^k q^{3k/2} T^k \Phi_0(h(2, k - 1))$$

$$\begin{aligned}
&= \frac{1-q^{-1}}{D_q} \sum_{i=1}^8 A_i U_i \sum_{k=1}^{\infty} \delta^k q^{-k/2} V_i^{k-1} \\
(144) \quad &= \frac{(1-q^{-1})q^{-1/2}\delta}{D_q} \sum_{i=1}^8 A_i \frac{U_i}{(1-\delta V_i q^{-1/2})}.
\end{aligned}$$

Case 2:

$$(145) \quad I_{1,2,T}(\eta) = \frac{(1-q^{-1})^2 q^{-3/2} \delta}{D_q} \sum_{i=1}^8 A_i \frac{U_i V_i}{(1-\delta V_i q^{-1/2})(1-q^{-2} U_i)}.$$

Case 3a:

$$(146) \quad I_{1,3a,T}(\eta) = \frac{(1-q^{-1})^2 q^{-3/2} \delta}{D_q} \sum_{i=1}^8 A_i \frac{U_i V_i (V_i + 1)}{(1-\delta V_i q^{-1/2})(1-q^{-1} V_i^2)}.$$

Case 3b:

$$(147) \quad I_{1,3b,T}(\eta) = \frac{(1-q^{-1})^2 (1-2q^{-1}) q^{-1/2} \delta}{D_q} \sum_{i=1}^8 A_i \frac{U_i V_i}{(1-\delta V_i q^{-1/2})(1-q^{-1} V_i^2)}.$$

Case 3c:

$$(148) \quad I_{1,3c,T}(\eta) = \frac{(1-q^{-1})^2 q^{-1/2} \delta}{D_q} \sum_{i=1}^8 A_i \frac{V_i^2}{(1-\delta V_i q^{-1/2})(1-q^{-1} V_i^2)}.$$

Case 4:

$$(149) \quad I_{1,4,T}(\eta) = \frac{(1-q^{-1})^3 q^{-3/2} \delta}{D_q} \sum_{i=1}^8 A_i \frac{U_i^2 V_i}{(1-\delta V_i q^{-1/2})(1-q^{-1} U_i)(1-q^{-1} V_i^2)}.$$

Case 5:

$$(150) \quad I_{1,5,T}(\eta) = \frac{(1-q^{-1})^3 q^{-5/2} \delta}{D_q} \sum_{i=1}^8 A_i \frac{U_i^2 V_i}{(1-\delta V_i q^{-1/2})(1-q^{-1} U_i)(1-q^{-2} U_i)}.$$

Case 6a:

$$(151) \quad I_{1,6a,T}(\eta) = \frac{(1-q^{-1})q^{-1}}{D_q} \sum_{i=1}^8 A_i V_i.$$

Case 6b:

$$(152) \quad I_{1,6b,T}(\eta) = \frac{(1-q^{-1})q^{-1/2}\delta^{-1}}{D_q} \sum_{i=1}^8 A_i \frac{U_i}{(1-\delta^{-1} V_i q^{-1/2})}.$$

Case 7:

$$(153) \quad I_{1,7,T}(\eta) = \frac{(1-q^{-1})^2 q^{-1}}{D_q} \sum_{i=1}^8 A_i \frac{U_i}{(1-\delta^{-1} V_i q^{-1/2})(1-q^{-2} U_i)}.$$

Case 8a:

$$(154) \quad I_{1,8a,T}(\eta) = \frac{(1-q^{-1})^2 q^{-1}}{D_q} \sum_{i=1}^8 A_i \frac{U_i (V_i + 1)}{(1-\delta^{-1} V_i q^{-1/2})(1-q^{-1} V_i^2)}.$$

Case 8b:

$$(155) \quad I_{1,8b,T}(\eta) = \frac{(1 - q^{-1})^2(1 - 2q^{-1})}{D_q} \sum_{i=1}^8 A_i \frac{U_i}{(1 - \delta^{-1}V_i q^{-1/2})(1 - q^{-1}V_i^2)}.$$

Case 8c:

$$(156) \quad I_{1,8c,T}(\eta) = \frac{(1 - q^{-1})^2}{D_q} \sum_{i=1}^8 A_i \frac{V_i}{(1 - \delta^{-1}V_i q^{-1/2})(1 - q^{-1}V_i^2)}.$$

Case 9:

$$(157) \quad I_{1,9,T}(\eta) = \frac{(1 - q^{-1})^3 q^{-1}}{D_q} \sum_{i=1}^8 A_i \frac{U_i^2}{(1 - \delta^{-1}V_i q^{-1/2})(1 - q^{-1}U_i)(1 - q^{-1}V_i^2)}.$$

Case 10:

$$(158) \quad I_{1,10,T}(\eta) = \frac{(1 - q^{-1})^3 q^{-2}}{D_q} \sum_{i=1}^8 A_i \frac{U_i^2}{(1 - \delta^{-1}V_i q^{-1/2})(1 - q^{-1}U_i)(1 - q^{-2}U_i)}.$$

2.10. **Calculation of $I_{2,T}(\eta)$.** By (80), we must evaluate

$$(159) \quad I_{2,T}(\eta) = -\frac{q}{1 - q^{-1}} \int_{F^\times} \int_F \int_F \eta(a) |a|^{3/2} \Phi_{0,T}(g(a; x, y, \varpi^{-2})) dx dy d^\times a.$$

The required special case of Lemma 2.12 is as follows.

Lemma 2.16. *Let $a \in F^\times$ and $x, y \in F$. Let*

$$(160) \quad g = g(a; x, y, \varpi^{-2}).$$

Define $\ell_1, \ell_2 \in \mathbb{Z}$ by

$$(161) \quad \ell_1 = \min\{-2, v(a), -v(a), v(ax), v(y), v(ay)\},$$

$$(162) \quad \ell_1 + \ell_2 = \min\{v(a) - 2, -v(a) - 2, v(ax), v(y), v(ay), v(a) + v(y^2 - x\varpi^{-2})\}.$$

Then $\ell_1 \leq \ell_2 \leq 0$ and

$$(163) \quad g \in Z(F)Kh(\ell, m)K \quad \text{with } \ell = -2\ell_2, m = \ell_2 - \ell_1.$$

We divide the integration domain into twenty-five regions. The last two columns give the indices ℓ and m in each region.

(164)

					ℓ	m
1a	$v(a) \geq 1$	$v(x) \geq -2v(a)$	$v(y) \geq -v(a)$	$v(a) \geq 2$	4	$v(a) - 2$
1b				$v(a) = 1$	2	1
2			$v(y) < -v(a)$		$-2v(y) - 2v(a)$	$v(a)$
3a		$v(x) < -2v(a)$	$v(y) \geq v(a) + v(x)$	$2v(y) \geq v(x) - 1$	4	$-v(a) - v(x) - 2$
3b				$2v(y) = v(x) - 2$	$\frac{y^2 \varpi^2}{x} - 1 \in \mathfrak{o}^\times$	$-v(a) - v(x) - 2$
3c					$\frac{y^2 \varpi^2}{x} - 1 \in \varpi \mathfrak{o}^\times$	$-v(a) - v(x) - 1$
3d					$\frac{y^2 \varpi^2}{x} - 1 \in \mathfrak{p}^2$	$-v(a) - v(x)$
4			$2v(y) \leq v(x) - 3$		$2v(x) - 4v(y)$	$2v(y) - 2v(x) - v(a)$
5			$v(y) < v(a) + v(x)$		$-2v(a) - 2v(y)$	$v(a)$
6a	$v(a) = 0$	$v(x) \geq 0$	$v(y) \geq -1$		0	2
6aa		$v(x) = -1$	$v(y) \geq -1$		2	1
6aaa		$v(x) \geq -1$	$v(y) \leq -2$		$-2v(y)$	0
6b	$v(a) = -1$	$v(x) \geq 0$	$v(y) \geq 0$		2	1
6bb		$v(x) = -1$	$v(y) \geq -1$		4	0
6bbb			$v(y) \leq -2$		$-2v(y)$	1
6c	$v(a) \leq -2$	$v(x) \geq 0$	$v(y) \geq 0$		4	$-v(a) - 2$
6cc		$v(x) = -1$	$v(y) \geq -1$		4	$-v(a) - 1$
6ccc			$v(y) \leq -2$		$-2v(y)$	$-v(a)$
7	$v(a) \leq -1$	$v(x) \geq 0$	$v(y) \leq -1$		$-2v(y)$	$-v(a)$
8a	$v(a) \leq 0$	$v(x) < -1$	$v(y) \geq v(x)$	$2v(y) \geq v(x) - 1$	4	$-v(a) - v(x) - 2$
8b				$2v(y) = v(x) - 2$	$\frac{y^2 \varpi^2}{x} - 1 \in \mathfrak{o}^\times$	$-v(a) - v(x) - 2$
8c					$\frac{y^2 \varpi^2}{x} - 1 \in \varpi \mathfrak{o}^\times$	$-v(a) - v(x) - 1$
8d					$\frac{y^2 \varpi^2}{x} - 1 \in \mathfrak{p}^2$	$-v(a) - v(x)$
9			$2v(y) \leq v(x) - 3$		$2v(x) - 4v(y)$	$2v(y) - 2v(x) - v(a)$
10			$v(y) < v(x)$		$-2v(y)$	$-v(a)$

Now we calculate all twenty-five pieces. We denote by $I_{2,j,T}(\eta)$ the piece of $I_{2,T}(\eta)$ corresponding to the j -th case. We show the full calculations only for the first case, the other cases being similar.

Case 1a: Here $k = v(a) \geq 2$, $(\ell, m) = (4, k - 2)$ and the height is k . Consequently,

$$\begin{aligned}
I_{2,1a,T}(\eta) &= -q \sum_{k=2}^{\infty} \delta^k q^{3k/2} T^k \Phi_0(h(4, k - 2)) \\
&= -\frac{q^{-1}}{D_q} \sum_{i=1}^8 A_i U_i^2 \sum_{k=2}^{\infty} \delta^k q^{-k/2} V_i^{k-2}
\end{aligned}$$

$$(165) \quad = -\frac{q^{-2}\delta^2}{D_q} \sum_{i=1}^8 A_i \frac{U_i^2}{(1 - \delta V_i q^{-1/2})}.$$

The other twenty-four regions are evaluated in the same way.

Case 1b:

$$(166) \quad I_{2,1b,T}(\eta) = -\frac{q^{-5/2}\delta}{D_q} \sum_{i=1}^8 A_i U_i V_i.$$

Case 2:

$$(167) \quad I_{2,2,T}(\eta) = -\frac{(1 - q^{-1})q^{-3/2}\delta}{D_q} \sum_{i=1}^8 A_i \frac{U_i V_i}{(1 - \delta V_i q^{-1/2})(1 - q^{-2}U_i)}.$$

Case 3a:

$$(168) \quad I_{2,3a,T}(\eta) = -\frac{(1 - q^{-1})q^{-3/2}\delta}{D_q} \sum_{i=1}^8 A_i \frac{U_i^2(1 + q^{-1}V_i)}{(1 - q^{-1}V_i^2)(1 - \delta V_i q^{-1/2})}.$$

Case 3b:

$$(169) \quad I_{2,3b,T}(\eta) = -\frac{(1 - q^{-1})(1 - 2q^{-1})q^{-3/2}\delta}{D_q} \sum_{i=1}^8 A_i \frac{U_i^2 V_i}{(1 - q^{-1}V_i^2)(1 - \delta V_i q^{-1/2})}.$$

Case 3c:

$$(170) \quad I_{2,3c,T}(\eta) = -\frac{(1 - q^{-1})^2 q^{-3/2}\delta}{D_q} \sum_{i=1}^8 A_i \frac{U_i V_i^2}{(1 - q^{-1}V_i^2)(1 - \delta V_i q^{-1/2})}.$$

Case 3d:

$$(171) \quad I_{2,3d,T}(\eta) = -\frac{(1 - q^{-1})q^{-3/2}\delta}{D_q} \sum_{i=1}^8 A_i \frac{V_i^3}{(1 - q^{-1}V_i^2)(1 - \delta V_i q^{-1/2})}.$$

Case 4:

$$(172) \quad I_{2,4,T}(\eta) = -\frac{(1 - q^{-1})^2 q^{-5/2}\delta}{D_q} \sum_{i=1}^8 A_i \frac{U_i^3 V_i}{(1 - \delta V_i q^{-1/2})(1 - q^{-1}V_i^2)(1 - q^{-1}U_i)}.$$

Case 5:

$$(173) \quad I_{2,5,T}(\eta) = -\frac{(1 - q^{-1})^2 q^{-5/2}\delta}{D_q} \sum_{i=1}^8 A_i \frac{U_i^2 V_i}{(1 - \delta V_i q^{-1/2})(1 - q^{-1}U_i)(1 - q^{-2}U_i)}.$$

Case 6a:

$$(174) \quad I_{2,6a,T}(\eta) = -\frac{q^{-2}}{D_q} \sum_{i=1}^8 A_i V_i^2.$$

Case 6aa:

$$(175) \quad I_{2,6aa,T}(\eta) = -\frac{(1 - q^{-1})q^{-2}}{D_q} \sum_{i=1}^8 A_i U_i V_i.$$

Case 6aaa:

$$(176) \quad I_{2,6aaa,T}(\eta) = -\frac{(1-q^{-1})q^{-2}}{D_q} \sum_{i=1}^8 A_i \frac{U_i^2}{(1-q^{-2}U_i)}.$$

Case 6b:

$$(177) \quad I_{2,6b,T}(\eta) = -\frac{q^{-5/2}\delta^{-1}}{D_q} \sum_{i=1}^8 A_i U_i V_i.$$

Case 6bb:

$$(178) \quad I_{2,6bb,T}(\eta) = -\frac{(1-q^{-1})q^{-3/2}\delta^{-1}}{D_q} \sum_{i=1}^8 A_i U_i^2.$$

Case 6bbb:

$$(179) \quad I_{2,6bbb,T}(\eta) = -\frac{(1-q^{-1})^2 q^{-5/2}\delta^{-1}}{D_q} \sum_{i=1}^8 A_i \frac{U_i^2 V_i}{(1-q^{-2}U_i)}.$$

Case 6c:

$$(180) \quad I_{2,6c,T}(\eta) = -\frac{q^{-2}\delta^{-2}}{D_q} \sum_{i=1}^8 A_i \frac{U_i^2}{(1-\delta^{-1}V_i q^{-1/2})}.$$

Case 6cc:

$$(181) \quad I_{2,6cc,T}(\eta) = -\frac{(1-q^{-1})q^{-2}\delta^{-2}}{D_q} \sum_{i=1}^8 A_i \frac{U_i^2 V_i}{(1-\delta^{-1}V_i q^{-1/2})}.$$

Case 6ccc:

$$(182) \quad I_{2,6ccc,T}(\eta) = -\frac{(1-q^{-1})^2 q^{-3}\delta^{-2}}{D_q} \sum_{i=1}^8 A_i \frac{U_i^2 V_i^2}{(1-\delta^{-1}V_i q^{-1/2})(1-q^{-2}U_i)}.$$

Case 7:

$$(183) \quad I_{2,7,T}(\eta) = -\frac{(1-q^{-1})q^{-3/2}\delta^{-1}}{D_q} \sum_{i=1}^8 A_i \frac{U_i V_i}{(1-\delta^{-1}V_i q^{-1/2})(1-q^{-2}U_i)}.$$

Case 8a:

$$(184) \quad I_{2,8a,T}(\eta) = -\frac{(1-q^{-1})q^{-2}}{D_q} \sum_{i=1}^8 A_i \frac{U_i^2 (V_i + 1)}{(1-\delta^{-1}V_i q^{-1/2})(1-q^{-1}V_i^2)}.$$

Case 8b:

$$(185) \quad I_{2,8b,T}(\eta) = -\frac{(1-q^{-1})(1-2q^{-1})q^{-1}}{D_q} \sum_{i=1}^8 A_i \frac{U_i^2}{(1-\delta^{-1}V_i q^{-1/2})(1-q^{-1}V_i^2)}.$$

Case 8c:

$$(186) \quad I_{2,8c,T}(\eta) = -\frac{(1-q^{-1})^2 q^{-1}}{D_q} \sum_{i=1}^8 A_i \frac{U_i V_i}{(1-\delta^{-1}V_i q^{-1/2})(1-q^{-1}V_i^2)}.$$

Case 8d:

$$(187) \quad I_{2,8d,T}(\eta) = -\frac{(1-q^{-1})q^{-1}}{D_q} \sum_{i=1}^8 A_i \frac{V_i^2}{(1-\delta^{-1}V_i q^{-1/2})(1-q^{-1}V_i^2)}.$$

Case 9:

$$(188) \quad I_{2,9,T}(\eta) = -\frac{(1-q^{-1})^2 q^{-2}}{D_q} \sum_{i=1}^8 A_i \frac{U_i^3}{(1-\delta^{-1}V_i q^{-1/2})(1-q^{-1}U_i)(1-q^{-1}V_i^2)}.$$

Case 10:

$$(189) \quad I_{2,10,T}(\eta) = -\frac{(1-q^{-1})^2 q^{-3}}{D_q} \sum_{i=1}^8 A_i \frac{U_i^3}{(1-\delta^{-1}V_i q^{-1/2})(1-q^{-1}U_i)(1-q^{-2}U_i)}.$$

2.11. Local L -factors and summary of calculations. In this subsection, we put together the calculations from the previous subsections and complete the proof of our main local theorem. Recall that the Satake parameters of π satisfy

$$|\alpha| = |\beta| = |\gamma| = 1, \quad \alpha\beta\gamma^2 = 1.$$

We retain the auxiliary character (see Section 2.7)

$$\eta = \eta_0 | \cdot |^{s_0}, \quad \eta_0 \text{ unitary and unramified}, \quad 0 \leq s_0 \leq \frac{1}{2}, \quad \delta = \eta(\varpi).$$

As before, we allow $|T| < 1$, and also $T = 1$ when $s_0 < \frac{1}{2}$. Recall D_q from (107) and $U_i(T) = T\gamma^2 L_i^2$, $V_i(T) = TM_i$ from (118). Put

$$\Delta_\delta = q^{-1/2}(\delta + \delta^{-1}),$$

and define the following polynomial in two variables U, V :

$$(190) \quad \begin{aligned} P_\delta(U, V) = & 1 + (1-q^{-2})V - q^{-1}(1+q^{-1} + (1-q^{-1})\Delta_\delta)V^2 \\ & + q^{-2}(\Delta_\delta + 1 - q^{-1})V^3 - q^{-3}V^4 \\ & + U \left\{ (1-q^{-1})(1+\Delta_\delta) - q^{-1}(\Delta_\delta + 2 - q^{-1} - q^{-2})V \right. \\ & \left. + q^{-2}(\Delta_\delta^2 + (1-q^{-1})\Delta_\delta + 1 - q^{-1})V^2 - q^{-3}(\Delta_\delta + 1 - q^{-1})V^3 \right\} \\ & + U^2 \left\{ -q^{-1}(\Delta_\delta^2 + (1-q^{-1})\Delta_\delta - 2q^{-1}) + q^{-2}(\Delta_\delta + 1 - q^{-1})V \right\}. \end{aligned}$$

The individual coefficients A_i in (109) are defined precisely when the Satake parameters of π satisfy

$$(191) \quad \alpha \neq 1, \quad \beta \neq 1, \quad \alpha \neq \beta, \quad \alpha\beta \neq 1.$$

We will assume (191) until the final continuity argument below. For any such Satake parameters, define the rational function

$$(192) \quad \mathcal{R}(\delta, T) := \frac{1}{D_q} \sum_{i=1}^8 A_i \frac{P_\delta(U_i(T), V_i(T))}{(1-\delta V_i(T)q^{-1/2})(1-\delta^{-1}V_i(T)q^{-1/2})}.$$

Adding the fifty region contributions in the preceding three subsections 2.8–2.10, and using (83) and (82), gives

$$(193) \quad \mathcal{R}(\delta, T) = \begin{cases} I_T(\eta), & 0 \leq s_0 \leq \frac{1}{2}, \quad |T| < 1, \\ I(\eta), & 0 \leq s_0 < \frac{1}{2}, \quad T = 1. \end{cases}$$

Now put

$$\begin{aligned} N_1 &= (1 - \alpha q^{-1})(1 - \alpha^{-1} q^{-1})(1 - \beta q^{-1})(1 - \beta^{-1} q^{-1}), \\ N_2 &= (1 - \alpha \beta q^{-1})(1 - \alpha^{-1} \beta q^{-1})(1 - \alpha \beta^{-1} q^{-1})(1 - \alpha^{-1} \beta^{-1} q^{-1}), \\ D_1(\delta) &= (1 - \alpha \delta q^{-1/2})(1 - \alpha^{-1} \delta q^{-1/2})(1 - \beta \delta q^{-1/2})(1 - \beta^{-1} \delta q^{-1/2}), \\ D_2(\delta) &= (1 - \alpha \delta^{-1} q^{-1/2})(1 - \alpha^{-1} \delta^{-1} q^{-1/2})(1 - \beta \delta^{-1} q^{-1/2})(1 - \beta^{-1} \delta^{-1} q^{-1/2}). \end{aligned}$$

Setting $T = 1$ in (192), summing over i with (109), using $\gamma^2 = (\alpha\beta)^{-1}$, and simplifying with the help of Mathematica (see also Remark 2.17) gives, for $0 \leq s_0 < \frac{1}{2}$,

$$(194) \quad I(\eta) = \mathcal{R}(\delta, 1) = \frac{(1 + \alpha)^2(1 + \beta)^2}{\alpha\beta} \frac{1 - q^{-1}}{D_q} \frac{N_1 N_2}{D_1(\delta) D_2(\delta)}.$$

Next take

$$\eta = \eta_{1/2} = (\cdot, \xi) | \cdot |^{1/2}, \quad \xi \in \mathfrak{o}^\times,$$

as required for the special representation $\sigma \simeq \tilde{\sigma}_\xi$. Thus $\eta_0 = (\cdot, \xi)$ is quadratic, and we put

$$\varepsilon_\sigma = (\varpi, \xi) = \eta_0(\varpi) \in \{\pm 1\}, \quad \delta_{\text{sp}} = \eta_{1/2}(\varpi) = \varepsilon_\sigma q^{-1/2}.$$

For either sign of ε_σ , define

$$D_{1,\text{sp}} := D_1(\delta_{\text{sp}}) = (1 - \alpha \varepsilon_\sigma q^{-1})(1 - \alpha^{-1} \varepsilon_\sigma q^{-1})(1 - \beta \varepsilon_\sigma q^{-1})(1 - \beta^{-1} \varepsilon_\sigma q^{-1}).$$

By (193), for $0 < T < 1$,

$$\frac{1 - \varepsilon_\sigma}{q + 1} I_T(\eta_{1/2}) = \frac{1 - \varepsilon_\sigma}{q + 1} \mathcal{R}(\delta_{\text{sp}}, T).$$

If $\varepsilon_\sigma = 1$, this complete product is zero for every $0 < T < 1$, so its limit as T approaches 1 is zero. Now let $\varepsilon_\sigma = -1$. Define

$$(195) \quad \begin{aligned} Q(U, V) &= 1 - q^{-2}V - q^{-3}V^2 - q^{-3}V^3 - (q^{-1} - q^{-2})U \\ &\quad + (q^{-2} + q^{-3})UV + 2q^{-4}UV^2 - 2q^{-3}U^2. \end{aligned}$$

Since $\Delta_{-q^{-1/2}} = -(1 + q^{-1})$, direct substitution in (190) gives the polynomial identity

$$(196) \quad P_{-q^{-1/2}}(U, V) = (1 + V)Q(U, V).$$

Consequently, for $|T| < 1$, (192) becomes

$$(197) \quad \mathcal{R}(-q^{-1/2}, T) = \frac{1}{D_q} \sum_{i=1}^8 A_i \frac{Q(U_i(T), V_i(T))}{1 + q^{-1}V_i(T)}.$$

Each factor $1 + q^{-1}V_i(T)$ has absolute value at least $1 - q^{-1}$ for $|T| \leq 1$. The right-hand side therefore extends continuously to $T = 1$, and its value there is the limit as $T \uparrow 1$.

Initially also assume $\alpha, \beta \neq -1$. The rational-function identity in (194) and the equality

$$D_2(-q^{-1/2}) = (1 + \alpha)(1 + \alpha^{-1})(1 + \beta)(1 + \beta^{-1}) = \frac{(1 + \alpha)^2(1 + \beta)^2}{\alpha\beta}$$

leads to cancellation in (194) and gives

$$\lim_{T \uparrow 1} \mathcal{R}(-q^{-1/2}, T) = \frac{1 - q^{-1}}{D_q} \frac{N_1 N_2}{D_{1,\text{sp}}}.$$

The expression at $T = 1$ on the right of (197) and the last displayed rational expression are continuous at all unitary parameters satisfying (191). The equality therefore holds also when

$\alpha = -1$ or $\beta = -1$; the additional restriction was needed only for the initial cancellation in (194). Combining this with (193) and the already established vanishing for $\varepsilon_\sigma = 1$, we obtain

$$(198) \quad \lim_{T \uparrow 1} \frac{1 - \varepsilon_\sigma}{q + 1} I_T(\eta_{1/2}) = \begin{cases} 0, & \varepsilon_\sigma = 1, \\ \frac{2(1 - q^{-1})}{(q + 1)D_q} \frac{N_1 N_2}{D_{1,\text{sp}}}, & \varepsilon_\sigma = -1. \end{cases}$$

Remark 2.17. *The calculations leading to the results (194) and (198) have been formalized and machine-verified in the Lean 4 proof assistant. More precisely, the formalized part begins with the core integrals $I(\eta)$ and $I_T(\eta)$ defined in (82) and (83), with the function Φ_0 defined directly by the formula in Proposition 2.11, and corresponds to the calculations in Sections 2.8–2.10 together with the relevant simplifications above leading to the conclusions (194) and (198). The Lean development is publicly available at*

<https://github.com//abhisaha-git/fourier-jacobi-core-evaluation>

The formalized statements assume that the parameters α, β satisfy (191) and that $\alpha, \beta \neq -1$. The formalization is also registered [1] in the Palomar Registry, a public, searchable registry of Lean formalizations that meet certain requirements.

We now write down the local L -factors in (32) and use the preceding computations to finish the proof. For π we have

$$(199) \quad L(1, \pi, \text{Ad})^{-1} = (1 - q^{-1})^2 N_1 N_2.$$

Recall also the standard degree-5 factor

$$(200) \quad L(s, \pi, \rho_5)^{-1} = (1 - q^{-s})(1 - \alpha q^{-s})(1 - \alpha^{-1} q^{-s})(1 - \beta q^{-s})(1 - \beta^{-1} q^{-s}).$$

We will use the identity

$$(201) \quad (1 - q^{-1})^2 D_q = (1 - q^{-2})(1 - q^{-4}).$$

Suppose first that $\sigma \simeq \tilde{\pi}(\chi)$ with χ unitary and unramified, and take $\eta = \chi$, so that $s_0 = 0$ and $\delta = \chi(\varpi)$. The Waldspurger lift of σ to $\text{PGL}_2(F)$ is the unramified principal series representation of $\text{PGL}_2(F)$ with Satake parameters δ and δ^{-1} ; see [37, Théorème 1]. Thus (33) and (200) give

$$(202) \quad \begin{aligned} L_\psi(s, \pi \times \sigma)^{-1} &= (1 - \delta q^{-s})(1 - \alpha \delta q^{-s})(1 - \alpha^{-1} \delta q^{-s})(1 - \beta \delta q^{-s})(1 - \beta^{-1} \delta q^{-s}) \\ &\times (1 - \delta^{-1} q^{-s})(1 - \alpha \delta^{-1} q^{-s})(1 - \alpha^{-1} \delta^{-1} q^{-s})(1 - \beta \delta^{-1} q^{-s})(1 - \beta^{-1} \delta^{-1} q^{-s}), \end{aligned}$$

and

$$(203) \quad L_\psi(s, \sigma, \text{Ad})^{-1} = (1 - \delta^2 q^{-s})(1 - q^{-s})(1 - \delta^{-2} q^{-s}).$$

By (85), we have $\alpha(\pi, \sigma; m) = 2I(\chi)/(q + 1)$. Substituting this and (194) into (32), and using (199), (202), (203) and (201), gives

$$(204) \quad \begin{aligned} \alpha^\#(\pi, \sigma; m) &= \frac{2(1 - q^{-2})(1 - q^{-4})}{q + 1} \frac{L(1, \pi, \text{Ad}) L_\psi(1, \sigma, \text{Ad})}{L_\psi(\frac{1}{2}, \pi \times \sigma)} I(\chi) \\ &= \frac{2}{q + 1} \frac{(\alpha^{-1/2} + \alpha^{1/2})^2 (\beta^{-1/2} + \beta^{1/2})^2}{(1 + \delta q^{-1/2})(1 + \delta^{-1} q^{-1/2})}. \end{aligned}$$

Here the squared expressions in the numerator equal $\alpha^{-1} + 2 + \alpha$ and $\beta^{-1} + 2 + \beta$, respectively, so the formula is independent of choices of square roots.

Next suppose that $\sigma \simeq \tilde{\sigma}_\xi$ with $\xi \in \mathfrak{o}^\times$ is a special representation, and retain $\eta_{1/2}$, ε_σ , δ_{sp} and $D_{1,\text{sp}}$ as above. Equation (89) identifies $\alpha(\pi, \sigma; m)$ with the complete limit evaluated in (198).

Hence

$$(205) \quad \alpha(\pi, \sigma; m) = \begin{cases} 0, & \varepsilon_\sigma = 1, \\ \frac{2(1-q^{-1})}{(q+1)D_q} \frac{N_1 N_2}{D_{1,\text{sp}}}, & \varepsilon_\sigma = -1. \end{cases}$$

The Waldspurger lift of σ is the Steinberg representation when $\varepsilon_\sigma = 1$, and its unramified quadratic twist when $\varepsilon_\sigma = -1$; see [37]. Accordingly, (33) and (200) give

$$(206) \quad L_\psi(s, \pi \times \sigma)^{-1} = (1 - \varepsilon_\sigma q^{-s-1/2})(1 - \alpha \varepsilon_\sigma q^{-s-1/2})(1 - \alpha^{-1} \varepsilon_\sigma q^{-s-1/2}) \\ \times (1 - \beta \varepsilon_\sigma q^{-s-1/2})(1 - \beta^{-1} \varepsilon_\sigma q^{-s-1/2}),$$

and

$$(207) \quad L_\psi(s, \sigma, \text{Ad})^{-1} = 1 - q^{-s-1}.$$

Using (199), (206), (207), (201) and (205) in (32), we obtain

$$(208) \quad \alpha^\#(\pi, \sigma; m) = \frac{1 - \varepsilon_\sigma}{q + 1} = \begin{cases} 0, & \varepsilon_\sigma = 1, \\ \frac{2}{q + 1}, & \varepsilon_\sigma = -1. \end{cases}$$

Finally, we remove (191) by a continuity argument. The “exceptional” Satake parameters are precisely those satisfying at least one of

$$\alpha = 1, \quad \beta = 1, \quad \alpha = \beta, \quad \alpha\beta = 1.$$

Any such Satake parameter can be approached by parameters satisfying (191). Lemma 2.3 gives a parameter-independent integrable majorant for each of the finitely many integrals in (36)–(37). Dominated convergence therefore proves continuity of the *original* local period $\alpha(\pi, \sigma; m)$ in terms of the Satake parameters of π , with σ fixed. By (199), (202), (203), (206) and (207), the normalizing L -factors in (32), at $s = 1$ or $s = \frac{1}{2}$ as appropriate, are finite, nonzero and continuous on the space of parameters. Thus $\alpha^\#(\pi, \sigma; m)$ is continuous there as well. The right-hand sides of (204) and (208) are also continuous, so the two identities extend to all unitary Satake parameters.

This completes the proof of Theorem 2.2.

3. GLOBAL RESULTS

In this final section, we return to a global setup. We first set the scene and prove some basic results on theta series, half-integral weight modular forms and their adelizations, and the classical interpretation of the adelic Fourier–Jacobi period for Siegel cusp forms. We then compute the archimedean factor in the refined Gan–Gross–Prasad conjecture in our setup. Finally, we prove our main global theorems and derive several applications.

3.1. Global notations. Let $\mathbb{A}$ denote the ring of adeles of $\mathbb{Q}$. Put $J_2 = \begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix}$. For every commutative ring R with identity, define

$$\text{GSp}_4(R) = \{g \in \text{GL}_4(R) : {}^t g J_2 g = \nu(g) J_2 \text{ for some } \nu(g) \in R^\times\},$$

and

$$G(R) = \text{Sp}_4(R) := \ker(\nu : \text{GSp}_4(R) \longrightarrow R^\times).$$

Let $H \subset G$ be the Heisenberg group defined by

$$H(R) = \left\{ h(\lambda, \mu, \kappa) := \begin{bmatrix} 1 & & \mu & \\ \lambda & 1 & \kappa & \\ & 1 & -\lambda & \\ & & & 1 \end{bmatrix} : \lambda, \mu, \kappa \in R \right\}.$$

We identify $h(\lambda, \mu, \kappa)$ with (λ, μ, κ) . In these coordinates,

$$(\lambda, \mu, \kappa)(\lambda', \mu', \kappa') = (\lambda + \lambda', \mu + \mu', \kappa + \kappa' + \lambda\mu' - \mu\lambda').$$

We identify SL_2 with the subgroup of G given by

$$\mathrm{SL}_2 \longrightarrow G, \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto \begin{bmatrix} a & b \\ c & 1 \\ & d & \\ & & 1 \end{bmatrix}.$$

The Jacobi group is

$$J = \mathrm{SL}_2 \cdot H \simeq \mathrm{SL}_2 \ltimes H.$$

We write $g(\lambda, \mu, \kappa)$ for

$$g h(\lambda, \mu, \kappa), \quad g \in \mathrm{SL}_2.$$

Let $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ be the metaplectic double cover of $\mathrm{SL}_2(\mathbb{A})$, and put

$$\widetilde{J}(\mathbb{A}) = \widetilde{\mathrm{SL}}_2(\mathbb{A}) \ltimes H(\mathbb{A}).$$

Note that the matrix realization of G , H , and J defined above differs from the one used in Section 2.

We write

$$\xi(s) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s)$$

for the completed Riemann zeta function. All our measures on adelic groups are the corresponding Tamagawa measures.

3.2. Theta series. Let $\mathcal{S}(\mathbb{A})$ denote the Schwartz space on $\mathbb{A}$. Its canonical inner product is

$$\langle \phi, \phi' \rangle := \int_{\mathbb{A}} \phi(x) \overline{\phi'(x)} dx,$$

where dx is the standard measure on $\mathbb{A}$.

Fix the standard character $\psi = \prod_v \psi_v : \mathbb{Q} \backslash \mathbb{A} \rightarrow \mathbb{C}^\times$, characterized by

$$\psi_\infty(x) = e^{2\pi i x}, \quad \mathrm{cond}(\psi_p) = \mathbb{Z}_p \quad (p < \infty).$$

Let m be a positive odd squarefree integer, and define

$$\psi^m(x) = \psi(mx).$$

Let $\phi^{(m)} = \bigotimes_v \phi_v^{(m)} \in \mathcal{S}(\mathbb{A})$, where

$$(209) \quad \phi_v^{(m)}(x) = \begin{cases} \mathbf{1}_{\mathbb{Z}_p}(x), & v = p < \infty, \ p \nmid 2m, \\ \mathbf{1}_{p^{-1}\mathbb{Z}_p}(x), & v = p < \infty, \ p \mid 2m, \\ e^{-2\pi m x^2}, & v = \infty. \end{cases}$$

A straightforward calculation gives

$$(210) \quad \langle \phi^{(m)}, \phi^{(m)} \rangle = \sqrt{m}.$$

Let ω_{ψ^m} be the Schrödinger–Weil (oscillator) representation of $\widetilde{J}(\mathbb{A}) = \widetilde{\mathrm{SL}}_2(\mathbb{A}) \ltimes H(\mathbb{A})$ with central character ψ^m , realized on the Schwartz space $\mathcal{S}(\mathbb{A})$. When convenient, we will consider ω_{ψ^m} a projective representation of $J(\mathbb{A})$. For $r \in \widetilde{J}(\mathbb{A})$, we define the theta function

$$\Theta_{\psi^m}(r, \phi^{(m)}) := \sum_{x \in \mathbb{Q}} (\omega_{\psi^m}(r) \phi^{(m)})(x).$$

For $\tau = x + iy \in \mathbb{H}$ and $z = u + iv \in \mathbb{C}$, set $g_{\tau, z} := \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} \begin{bmatrix} \sqrt{y} & \\ & 1/\sqrt{y} \end{bmatrix} \begin{pmatrix} v/\sqrt{y} & u/\sqrt{y} & 0 \end{pmatrix} \in J(\mathbb{R})$. We use the same notation $g_{\tau, z}$ for the element of $\widetilde{J}(\mathbb{A})$ whose finite components are trivial and whose

archimedean component is the lift of $g_{\tau,z}$ obtained by taking the sign $+1$ on its $\mathrm{SL}_2(\mathbb{R})$ -component. Recall the classical theta series on $\mathbb{H} \times \mathbb{C}$:

$$(211) \quad \theta_m(\tau, z) = \sum_{r \in \mathbb{Z}} e^{\frac{2\pi i r^2 \tau}{4m}} e^{2\pi i r z} = \sum_{\alpha \pmod{2m}} \theta_{m,\alpha}(\tau, z).$$

where

$$(212) \quad \theta_{m,\alpha}(\tau, z) = \sum_{r \equiv \alpha \pmod{2m}} e^{\frac{2\pi i r^2 \tau}{4m}} e^{2\pi i r z}.$$

For $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{R})$, $(\lambda, \mu, \kappa) \in H(\mathbb{R})$ and $(\tau, z) \in \mathbb{H} \times \mathbb{C}$ define the automorphy factor by

$$(213) \quad j_{\frac{1}{2},m}(\begin{bmatrix} a & b \\ c & d \end{bmatrix}(\lambda, \mu, \kappa), (\tau, z)) = \exp\left(-2\pi i m(\kappa - \frac{c(z - \lambda\tau + \mu)^2}{c\tau + d} + \lambda^2\tau + 2\lambda z + \lambda\mu)\right) j_{\frac{1}{2}}(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tau),$$

where

$$j_{\frac{1}{2}}(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tau) = \begin{cases} \sqrt{d} & \text{if } c = 0, d > 0; \\ -\sqrt{d} & \text{if } c = 0, d < 0; \\ (c\tau + d)^{\frac{1}{2}} & \text{if } c \neq 0. \end{cases}$$

An easy calculation shows that

$$j_{\frac{1}{2},m}(g_{\tau,z}, (i, 0)) = y^{-1/4} e^{-2\pi i m(\frac{uv}{y} + i\frac{v^2}{y})}.$$

Proposition 3.1. *With the above notations,*

$$\Theta_{\psi^m}(g_{\tau,z}, \phi^{(m)}) = j_{\frac{1}{2},m}(g_{\tau,z}, (i, 0))^{-1} \theta_m(\tau, z).$$

Proof. We first recall the relevant formulas for the oscillator representation at the archimedean place. Let

$$n(b) = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}, \quad a(t) = \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}, \quad t > 0.$$

For $f \in \mathcal{S}(\mathbb{R})$ and $\xi \in \mathbb{R}$,

$$(214) \quad \omega_{\psi_\infty^m}(n(b))f(\xi) = \psi_\infty^m(b\xi^2)f(\xi) = e^{2\pi i m b \xi^2} f(\xi),$$

$$(215) \quad \omega_{\psi_\infty^m}(a(t))f(\xi) = t^{1/2} f(t\xi),$$

and, for $(\lambda, \mu, \kappa) \in H(\mathbb{R})$,

$$\omega_{\psi_\infty^m}((\lambda, \mu, \kappa))f(\xi) = \psi_\infty^m(\kappa + \lambda\mu + 2\mu\xi)f(\xi + \lambda) = e^{2\pi i m(\kappa + \lambda\mu + 2\mu\xi)} f(\xi + \lambda).$$

Now, let $\eta \in \mathbb{Q}$. We have

$$\begin{aligned} \omega_{\psi_\infty^m}(g_{\tau,z})\phi_\infty^{(m)}(\eta) &= \omega_{\psi_\infty^m}\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{y} & \\ & 1/\sqrt{y} \end{bmatrix} \left(\frac{v}{\sqrt{y}}, \frac{u}{\sqrt{y}}, 0\right)\right) \phi_\infty^{(m)}(\eta) \\ &= \psi_\infty^m(x\eta^2) \omega_{\psi_\infty^m}\left(\begin{bmatrix} \sqrt{y} & \\ & 1/\sqrt{y} \end{bmatrix} \left(\frac{v}{\sqrt{y}}, \frac{u}{\sqrt{y}}, 0\right)\right) \phi_\infty^{(m)}(\eta) \\ &= y^{1/4} \psi_\infty^m(x\eta^2) \psi_\infty^m(2\eta u + \frac{vu}{y}) \phi_\infty^{(m)}(\sqrt{y}\eta + \frac{v}{\sqrt{y}}) \\ &= y^{1/4} e^{2\pi i m(\frac{vu}{y} + i\frac{v^2}{y})} e^{2\pi i m \tau \eta^2} e^{2\pi i m z (2\eta)} \\ &= j_{\frac{1}{2},m}(g_{\tau,z}, (i, 0))^{-1} e^{2\pi i m \tau \eta^2} e^{2\pi i m z (2\eta)}. \end{aligned}$$

Note that, for $\eta \in \mathbb{Q}$ we have

$$\phi_p^{(m)}(\eta) \neq 0 \Leftrightarrow \eta \in \begin{cases} \mathbb{Z}_p & \text{if } p \nmid 2m; \\ \frac{1}{p}\mathbb{Z}_p & \text{if } p \mid 2m. \end{cases}$$

Hence

$$\begin{aligned}
\Theta_{\psi^m}(g_{\tau,z}, \phi^{(m)}) &= j_{\frac{1}{2},m}(g_{\tau,z}, (i, 0))^{-1} \sum_{\eta \in \frac{1}{2m}\mathbb{Z}} e^{2\pi i m \tau \eta^2} e^{2\pi i m z(2\eta)} \\
&= j_{\frac{1}{2},m}(g_{\tau,z}, (i, 0))^{-1} \sum_{\eta \in \mathbb{Z}} e^{\frac{2\pi i \tau \eta^2}{4m}} e^{2\pi i z \eta} \\
&= j_{\frac{1}{2},m}(g_{\tau,z}, (i, 0))^{-1} \theta_m(\tau, z),
\end{aligned}$$

as required. $\square$

3.3. Siegel modular forms and representations. For the rest of the paper, fix an even integer $k \geq 4$. We recall some well-known properties of Siegel modular forms of degree 2; we refer the reader to [3] for proofs and further details. Let $S_k(\mathrm{Sp}_4(\mathbb{Z}))$ denote the space of Siegel cusp forms of weight k and full level. For any $F \in S_k(\mathrm{Sp}_4(\mathbb{Z}))$, define the Petersson inner product

$$(216) \quad \langle F, F \rangle = \int_{\mathrm{Sp}_4(\mathbb{Z}) \backslash \mathbb{H}_2} |F(Z)|^2 (\det Y)^{k-3} dX dY.$$

Recall that $G = \mathrm{Sp}_4$. For any $F \in S_k(\mathrm{Sp}_4(\mathbb{Z}))$, we let $\Psi_F \in L^2(G(\mathbb{Q}) \backslash G(\mathbb{A}))$ be the automorphic form obtained by adelizing F . We define $\langle \Psi_F, \Psi_F \rangle = \int_{G(\mathbb{Q}) \backslash G(\mathbb{A})} |\Psi_F(g)|^2 dg$, where we use the Tamagawa measure. By a routine calculation we have

$$(217) \quad \frac{\langle F, F \rangle}{2\xi(2)\xi(4)} = \frac{\langle F, F \rangle}{\mathrm{vol}(\mathrm{Sp}_4(\mathbb{Z}) \backslash \mathbb{H}_2)} = \frac{\langle \Psi_F, \Psi_F \rangle}{\mathrm{vol}(G(\mathbb{Q}) \backslash G(\mathbb{A}))} = \langle \Psi_F, \Psi_F \rangle.$$

Suppose that $F \in S_k(\mathrm{Sp}_4(\mathbb{Z}))$ is an eigenfunction of the local Hecke algebra at all finite primes. Then, by Theorem 3.1 of [26], Ψ_F generates an irreducible cuspidal automorphic representation π'_F of $G(\mathbb{A})$. Furthermore, Ψ_F extends to an automorphic form on $\mathrm{GSp}_4(\mathbb{A})$ of trivial central character, which generates an irreducible automorphic cuspidal representation π_F of $\mathrm{GSp}_4(\mathbb{A})$. For our purposes, it makes little difference whether we work with π'_F or π_F because it will not change the periods or the L -functions relevant for us.

We have an orthogonal (with respect to Petersson inner product) direct sum decomposition

$$S_k(\mathrm{Sp}_4(\mathbb{Z})) = S_k(\mathrm{Sp}_4(\mathbb{Z}))^{\mathrm{SK}} \oplus S_k(\mathrm{Sp}_4(\mathbb{Z}))^{\mathrm{T}}$$

where $S_k(\mathrm{Sp}_4(\mathbb{Z}))^{\mathrm{SK}}$ is the span of the Saito–Kurokawa lifts. In the sequel we will assume that our forms F are Hecke eigenforms lying in $S_k(\mathrm{Sp}_4(\mathbb{Z}))^{\mathrm{T}}$; by a deep theorem of Weissauer [38], this means that π_F is tempered at all primes.

Each $F \in S_k(\mathrm{Sp}_4(\mathbb{Z}))$ has the Fourier expansion

$$F(Z) = \sum_S a(F, S) e^{2\pi i \mathrm{Tr}(SZ)}$$

where the Fourier coefficients $a(F, S)$ are indexed by matrices S of the form

$$(218) \quad S = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}, \quad a, b, c \in \mathbb{Z}, \quad a > 0, \quad \mathrm{disc}(S) := b^2 - 4ac < 0,$$

and the Fourier–Jacobi expansion

$$(219) \quad F(Z) = \sum_{r>0} f_r(\tau, z) e^{2\pi i r \tau'}.$$

Here $Z = \begin{bmatrix} \tau & z \\ z & \tau' \end{bmatrix}$.

3.4. Half-integral weight forms and their adelizations. For a finite prime p and $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Q}_p)$ define

$$(220) \quad s_p\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{cases} (c, d)_p & \text{if } cd \neq 0 \text{ and } v_p(c) \text{ is odd,} \\ 1 & \text{otherwise,} \end{cases}$$

where $(\cdot, \cdot)_p$ is the local Hilbert symbol. For $g = (g_p)_p \in \mathrm{SL}_2(\mathbb{Q})$ let

$$(221) \quad s_{\mathbb{A}}(g) = \prod_{p < \infty} s_p(g).$$

By [14, Proposition 2.15] the map $i : \mathrm{SL}_2(\mathbb{Q}) \rightarrow \widetilde{\mathrm{SL}}_2(\mathbb{A})$ given by

$$i(g) = (g, s_{\mathbb{A}}(g))$$

is a homomorphism. Via i we consider $\mathrm{SL}_2(\mathbb{Q})$ a subgroup of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$.

For $\xi \in \mathbb{Q}^\times$ and $g \in \widetilde{\mathrm{SL}}_2(\mathbb{A})$, we define g^ξ by the global equivalent of (18).

For $(g, \varepsilon) \in \widetilde{\mathrm{SL}}_2(\mathbb{R})$ with $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\varepsilon \in \{\pm 1\}$, $z \in \mathbb{H}$, define the metaplectic automorphy factor by

$$J((g, \varepsilon), z) = \varepsilon \sqrt[4]{cz + d},$$

where the branch of the square root is $\sqrt[4]{re^{i\theta}} = \sqrt{r}e^{i\theta/2}$ for $r > 0$ and $\theta \in [-\pi, \pi)$.

Let Γ be a finite-index subgroup of either $\Gamma_0(4)$ or $\Gamma^0(4)$. For

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma, \quad z \in \mathbb{H},$$

define the classical half-integral weight automorphy factor [34, 35] by

$$(222) \quad \tilde{j}(g, z) = \left(\frac{c}{d}\right) \varepsilon_d^{-1} \sqrt{cz + d}$$

where now the branch of the square root is $\sqrt{re^{i\theta}} = \sqrt{r}e^{i\theta/2}$ for $r > 0$ and $\theta \in (-\pi, \pi]$, and

$$\varepsilon_d = \begin{cases} 1 & \text{if } d \equiv 1 \pmod{4} \\ i & \text{if } d \equiv 3 \pmod{4} \end{cases}.$$

For a holomorphic function $f : \mathbb{H} \rightarrow \mathbb{C}$ and $g \in \Gamma$, denote

$$(f|_{k-\frac{1}{2}}g)(\tau) = \tilde{j}(g, \tau)^{-2k+1} f(g\tau).$$

When Γ is contained in $\Gamma_0(N)$ or $\Gamma^0(N)$, and χ is a Dirichlet character modulo N , we regard χ as a character of Γ by $\chi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \chi(d)$. For any unitary character χ of Γ , we define $M_{k-\frac{1}{2}}(\Gamma, \chi)$ to be the space of holomorphic functions $f : \mathbb{H} \rightarrow \mathbb{C}$ satisfying

$$f|_{k-\frac{1}{2}}\gamma = \chi(\gamma)f \quad \text{for all } \gamma \in \Gamma,$$

and which are holomorphic at every cusp of Γ .

We define $S_{k-\frac{1}{2}}(\Gamma, \chi)$ to be the subspace of $M_{k-\frac{1}{2}}(\Gamma, \chi)$ consisting of those forms which vanish at every cusp. When χ is the trivial character, we write

$$M_{k-\frac{1}{2}}(\Gamma) = M_{k-\frac{1}{2}}(\Gamma, \mathbf{1}), \quad S_{k-\frac{1}{2}}(\Gamma) = S_{k-\frac{1}{2}}(\Gamma, \mathbf{1}).$$

For $h_1, h_2 \in S_{k-\frac{1}{2}}(\Gamma, \chi)$, the Petersson inner product is normalized as follows:

$$\langle h_1, h_2 \rangle = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma]^{-1} \int_{\Gamma \backslash \mathbb{H}} h_1(z) \overline{h_2(z)} y^{k-\frac{1}{2}} \frac{dx dy}{y^2}.$$

Given $h \in S_{k-\frac{1}{2}}(\Gamma, \chi)$, we say that an automorphic form $\Lambda_h \in L^2(\mathrm{SL}_2(\mathbb{Q}) \backslash \widetilde{\mathrm{SL}}_2(\mathbb{A}))$ is the adelization of h if

$$(223) \quad \Lambda_h(i(\gamma)(g_\infty, \varepsilon)) = J((g_\infty, \varepsilon), i)^{-2k+1} h(g_\infty i)$$

for $\gamma \in \mathrm{SL}_2(\mathbb{Q})$, $(g_\infty, \varepsilon) \in \widetilde{\mathrm{SL}}_2(\mathbb{R})$. Because $i(\mathrm{SL}_2(\mathbb{Q}))\widetilde{\mathrm{SL}}_2(\mathbb{R})$ is dense in $\widetilde{\mathrm{SL}}_2(\mathbb{A})$, each $h \in S_{k-\frac{1}{2}}(\Gamma, \chi)$ has a unique adelization Λ_h . Assuming that $\Gamma = \Gamma^0(N)$ (resp. $\Gamma = \Gamma_0(N)$) with $N = \prod p^{n_p}$, the adelization has the property that, for odd primes p and for $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma^0(p^{n_p}\mathbb{Z}_p)$ (resp. $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(p^{n_p}\mathbb{Z}_p)$) and $\varepsilon \in \{\pm 1\}$,

$$(224) \quad \Lambda_h(g(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \varepsilon)) = \varepsilon \chi_p(a) \Lambda_h(g)$$

for all $g \in \widetilde{\mathrm{SL}}_2(\mathbb{A})$. Here, χ_p is the local component of the adelization of the Dirichlet character χ . (There is an analogous property for $p = 2$, which we will not need.)

It is easy to check that if $h \in S_{k-\frac{1}{2}}(\Gamma, \chi)$ has adelization Λ_h , then

$$(225) \quad \frac{\langle h, h \rangle}{\pi/3} = \frac{\langle h, h \rangle}{\mathrm{vol}(\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H})} = \frac{\langle \Lambda_h, \Lambda_h \rangle}{\mathrm{vol}(\mathrm{SL}_2(\mathbb{Q}) \backslash \mathrm{SL}_2(\mathbb{A}))} = \langle \Lambda_h, \Lambda_h \rangle.$$

Recall here that

$$\langle \Lambda_h, \Lambda_h \rangle = \int_{\mathrm{SL}_2(\mathbb{Q}) \backslash \mathrm{SL}_2(\mathbb{A})} |\Lambda_h(g)|^2 dg,$$

and that all our adelic groups are equipped with the Tamagawa measure.

Now, let m be an odd, positive, squarefree integer. We recall that the Kohnen plus space $S_{k-\frac{1}{2}}^+(\Gamma_0(4m))$ is the subspace of $S_{k-\frac{1}{2}}(\Gamma_0(4m))$ consisting of those forms $h(z) = \sum_{n \geq 1} a_h(n) e^{2\pi i n z}$ such that $a_h(n) = 0$ unless $n \equiv 0, 3 \pmod{4}$ (recall that k is even). Similarly, let $S_{k-\frac{1}{2}}^+(\Gamma^0(4m), \chi_m)$ be the subspace of $S_{k-\frac{1}{2}}(\Gamma^0(4m), \chi_m)$ consisting of forms $h(z) = \sum_{n \geq 1} a_h(n) e^{2\pi i n z / (4m)}$ such that $a_h(n) = 0$ unless $n \equiv 0, 3 \pmod{4}$.

Let $h \in S_{k-\frac{1}{2}}^+(\Gamma_0(4m))$. Suppose that h' is given by $h'(\tau) = h(\frac{\tau}{4m})$. The following lemma is the analog of the local fact (21).

Lemma 3.2. *We have*

$$h' \in S_{k-\frac{1}{2}}^+(\Gamma^0(4m), \chi_m),$$

where χ_m is the Dirichlet character mod $4m$ defined by $\chi_m(a) = \left(\frac{m}{a}\right)$ for an integer relatively prime to $4m$.

Proof. Let $\delta = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma^0(4m)$, and put $\delta^\# = \begin{bmatrix} a & b/4m \\ 4mc & d \end{bmatrix} \in \Gamma_0(4m)$. Then

$$h'(\delta\tau) = h(\delta^\#(\tau/4m)) = \tilde{j}(\delta^\#, \tau/4m)^{2k-1} h(\tau/4m) = \tilde{j}(\delta^\#, \tau/4m)^{2k-1} h'(\tau)$$

and

$$\tilde{j}(\delta^\#, \tau/4m) = \varepsilon_d^{-1} \left(\frac{4mc}{d}\right) (c\tau + d)^{1/2} = \chi_m(d) \tilde{j}(\delta, \tau),$$

since d is odd. So $h'|_{k-\frac{1}{2}}\delta = \chi_m(d)h'$. The fact that h' has the required vanishing of Fourier coefficients is immediate. $\square$

Let Λ_h be the adelization of h . Put

$$\beta = \begin{bmatrix} (4m)^{-1/2} & 0 \\ 0 & (4m)^{1/2} \end{bmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

Lemma 3.3. *The adelization $\Lambda_{h'}$ of h' is given by*

$$(226) \quad \Lambda_{h'}(g) = (4m)^{\frac{k}{2}-\frac{1}{4}} \Lambda_h(g^{(4m)^{-1}}(\beta, 1))$$

for $g \in \widetilde{\mathrm{SL}}_2(\mathbb{A})$.

Proof. Let Λ' be the function on the right hand side of (226). It suffices to check $\Lambda_{h'} = \Lambda'$ on $i(\mathrm{SL}_2(\mathbb{Q}))\widetilde{\mathrm{SL}}_2(\mathbb{R})$. Since m is positive, twisting by $(4m)^{-1}$ is an automorphism of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ preserving $\mathrm{SL}_2(\mathbb{Q})$. Hence, for $g = i(\gamma)(g_\infty, \varepsilon)$,

$$\Lambda'(g) = (4m)^{\frac{k}{2}-\frac{1}{4}} \Lambda_h((g_\infty, \varepsilon)^{(4m)^{-1}}(\beta, 1)).$$

We have

$$(g_\infty, \varepsilon)^{(4m)^{-1}}(\beta, 1) = (\beta g_\infty, \varepsilon).$$

Hence

$$\Lambda'(g) = (4m)^{\frac{k}{2}-\frac{1}{4}} J((\beta g_\infty, \varepsilon), i)^{-(2k-1)} h(\beta g_\infty i).$$

Since

$$\beta g_\infty i = \frac{g_\infty i}{4m}, \quad J((\beta g_\infty, \varepsilon), i) = (4m)^{1/4} J((g_\infty, \varepsilon), i),$$

this becomes

$$\Lambda'(g) = J((g_\infty, \varepsilon), i)^{-(2k-1)} h'(g_\infty i).$$

By (223), the right hand side equals $\Lambda_{h'}(g_\infty, \varepsilon) = \Lambda_{h'}(g)$, proving the lemma. $\square$

For later use, we note the following relation of Petersson norms, which follows from an elementary computation, together with (225).

$$(227) \quad \frac{\pi}{3} \langle \Lambda_{h'}, \Lambda_{h'} \rangle = \langle h', h' \rangle = (4m)^{k-1/2} \langle h, h \rangle = \frac{\pi}{3} (4m)^{k-1/2} \langle \Lambda_h, \Lambda_h \rangle.$$

Now suppose that Λ_h generates an irreducible genuine automorphic representation σ_h of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$. In (19) we defined the twist of a local representation by a non-zero element of the local field. The twist of a global representation by a non-zero rational element is defined analogously. It follows from Lemma 3.3 that $\Lambda_{h'}$ generates the irreducible automorphic representation

$$(228) \quad \sigma_h^{(m)} := (m^{-1}) \cdot \sigma_h \simeq m \cdot \sigma_h$$

of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$. It follows from (224) that, for each prime $p \mid m$,

$$\Lambda_{h'}(g(\gamma_p, \varepsilon)) = \varepsilon \chi_{m,p}(a) \Lambda_{h'}(g), \quad \gamma_p = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma^0(m\mathbb{Z}_p), \quad \varepsilon \in \{\pm 1\},$$

for $g \in \widetilde{\mathrm{SL}}_2(\mathbb{A})$. Here, $\chi_{m,p} = (\cdot, m)_p$ is the unique nontrivial quadratic character of $\mathbb{Z}_p^\times$. Note that p is odd because m is.

For any non-trivial global character ψ' of $\mathbb{Q} \backslash \mathbb{A}$, Waldspurger has defined a lifting $\mathrm{Wald}_{\psi'}$ from the set of automorphic representations of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ that are not “elementary theta series” to the set of automorphic representations of $\mathrm{GL}_2(\mathbb{A})$ with trivial central character; see [37, Section VI.2]. The global lifting is compatible with the local liftings considered earlier.

The representation generated by $\overline{\Lambda_h}$ is the complex conjugate representation $\overline{\sigma_h}$, which, since σ_h is unitary, is isomorphic to the contragredient representation $\sigma_h^\vee$:

$$\overline{\sigma_h} \simeq \sigma_h^\vee \simeq (-1) \cdot \sigma_h.$$

The next lemma explains the relationship between the Waldspurger lifting of σ_h and the classical Shimura lift of h .

Lemma 3.4. *Let σ_h be the genuine automorphic representation generated by the adelization Λ_h of $h \in S_{k-\frac{1}{2}}^+(\Gamma_0(4m))$, and let $\pi_{0,h}$ be the automorphic representation of $\mathrm{PGL}_2(\mathbb{A})$ generated by a classical Shimura lift of h . Then*

$$(229) \quad \pi_{0,h} \simeq \mathrm{Wald}_{\psi}(\overline{\sigma_h}) \simeq \mathrm{Wald}_{\psi^{-1}}(\sigma_h).$$

Proof. This follows from the calculations in [5]. More precisely, the character denoted there by ψ_S is $\psi_S(x) = \psi((-1)^{k-1}x) = \psi^{-1}(x)$, since k is even, and [5, Section 10.6] gives

$$\pi_{0,h} \simeq \text{Wald}_{\psi^{-1}}(\sigma_h).$$

Since $\pi_{0,h}$ is unitary and has trivial central character, we also have

$$\text{Wald}_{\psi}(\overline{\sigma_h}) \simeq \overline{\text{Wald}_{\psi^{-1}}(\sigma_h)} \simeq \pi_{0,h}.$$

□

Properties of the local theta correspondence give $\text{Wald}_{\psi_p^m}(m \cdot \rho_p) = \text{Wald}_{\psi_p}(\rho_p)$ for any irreducible admissible representation ρ_p of $\widetilde{\text{SL}}_2(\mathbb{Q}_p)$ that is not an even Weil representation. It therefore follows from (228), (229), and strong multiplicity one for GL_2 that

$$(230) \quad \text{Wald}_{\psi^m}(\overline{\sigma_h^{(m)}}) = \text{Wald}_{\psi}(\overline{\sigma_h}) = \pi_{0,h}.$$

We say that a genuine irreducible cuspidal automorphic representation σ of $\widetilde{\text{SL}}_2(\mathbb{A})$ occurs in $S_{k-\frac{1}{2}}^+(\Gamma_0(4m))$ if there exists a cusp form in $S_{k-\frac{1}{2}}^+(\Gamma_0(4m))$ whose adelization generates σ . We refer to the span of all such cusp forms as the σ -isotypic subspace of $S_{k-\frac{1}{2}}^+(\Gamma_0(4m))$ and denote this subspace by $\mathcal{V}_{\sigma}$. We have a direct sum decomposition

$$S_{k-\frac{1}{2}}^+(\Gamma_0(4m)) = \bigoplus_{\sigma} \mathcal{V}_{\sigma},$$

where σ ranges over the (distinct) representations occurring in $S_{k-\frac{1}{2}}^+(\Gamma_0(4m))$. Given an element h of $S_{k-\frac{1}{2}}^+(\Gamma_0(4m))$, it lies inside a single $\mathcal{V}_{\sigma}$ if and only if it is an eigenform for the Hecke operators $T(n^2)$ for all $(n, 4m) = 1$.

3.5. Newforms and orthogonal Hecke bases of $S_{k-\frac{1}{2}}^+(\Gamma_0(4m))$. For each genuine cuspidal automorphic representation σ of $\widetilde{\text{SL}}_2(\mathbb{A})$ that occurs in $S_{k-\frac{1}{2}}^+(\Gamma_0(4m))$, we now construct an orthogonal basis $\mathcal{B}_{\sigma}$ for the subspace $\mathcal{V}_{\sigma}$ of $S_{k-\frac{1}{2}}^+(\Gamma_0(4m))$.

Let $\sigma \cong \otimes \sigma_p$, where σ_p is an irreducible, admissible representation of $\widetilde{\text{SL}}_2(\mathbb{Q}_p)$. Note that σ_{∞} is a holomorphic discrete series representation of lowest weight $k - \frac{1}{2}$, that σ_p is an unramified principal series representation at all finite primes not dividing m , and that at the primes dividing m it is either an unramified principal series representation or a special representation. We fix distinguished vectors at each place, as follows. Recall the spaces $V_{\sigma_p}(\Gamma, \chi)$ defined by the condition (20).

- At the archimedean place, let v_{∞} be a vector of weight $k - 1/2$.
- At places $p \mid m$ for which σ_p is an unramified principal series representation, let $v_p^{(1)}$ and $v_p^{(2)}$ be an orthogonal basis of the 2-dimensional space $V_{\sigma_p}(\Gamma_0(p\mathbb{Z}_p), 1)$.
- At places $p \mid m$ for which σ_p is a special representation, let $v_p^{(1)}$ be a vector spanning the 1-dimensional space $V_{\sigma_p}(\Gamma_0(p\mathbb{Z}_p), 1)$.
- At odd primes $p \nmid m$, let $v_p^{(1)}$ be a vector spanning the 1-dimensional space $V_{\sigma_p}(\text{SL}_2(\mathbb{Z}_p), 1)$.
- For $p = 2$ the definition of the spaces $V_{\sigma_p}(\Gamma, \chi)$ has to be modified. Let $V_{\sigma_2}(\Gamma_0(4\mathbb{Z}_2), 1)$ be the space of vectors v in the space of σ_2 with the following properties:

$$\begin{aligned} \sigma_2\left(\begin{bmatrix} 1 & b \\ & 1 \end{bmatrix}, \varepsilon\right)v &= \varepsilon v & \text{for } b \in \mathbb{Z}_2, \\ \sigma_2\left(\begin{bmatrix} a & \\ & a^{-1} \end{bmatrix}, \varepsilon\right)v &= \varepsilon \delta_{\psi_2}(a)v & \text{for } a \in \mathbb{Z}_2^{\times}, \\ \sigma_2\left(\begin{bmatrix} 1 & \\ c & 1 \end{bmatrix}, \varepsilon\right)v &= \varepsilon v & \text{for } c \in 4\mathbb{Z}_2, \end{aligned}$$

Here, $\delta_{\psi_2}(a)$ is the Weil constant. One can show by calculations in the induced models that $\dim V_{\sigma_2}(\Gamma_0(4\mathbb{Z}_2), 1) = 2$. The plus space condition singles out a 1-dimensional subspace of $V_{\sigma_2}(\Gamma_0(4\mathbb{Z}_2), 1)$, and we let $v_2^{(1)}$ be a vector spanning this subspace.

Let $C(\sigma)$ be the product of all the primes p where σ_p is special; note that $C(\sigma)$ divides m . For each positive integer i such that i divides $\frac{m}{C(\sigma)}$, we define a vector $v_\sigma^{(i)}$ in the space of $\otimes'_{v < \infty} \sigma_v$ by

$$v_\sigma^{(i)} = \left(\otimes_{p|i} v_{\sigma_p}^{(2)} \right) \otimes \left(\otimes_{p \nmid i} v_{\sigma_p}^{(1)} \right).$$

For each such $v_\sigma^{(i)}$, let $\Lambda_{\sigma,i}$ be the automorphic form in the space of σ corresponding to $v_\infty \otimes v_\sigma^{(i)}$. By de-adelization, there is a unique cusp form $h_{\sigma,i} \in S_{k-\frac{1}{2}}^+(\Gamma_0(4m))$ whose adelization is $\Lambda_{\sigma,i}$. Define

$$(231) \quad \mathcal{B}_\sigma = \left\{ h_{\sigma,i} : i \geq 1, i \mid \frac{m}{C(\sigma)} \right\}.$$

Then $\mathcal{B}_\sigma$ is a set of vectors in $S_{k-\frac{1}{2}}^+(\Gamma_0(4m))$ that form an orthogonal basis of $\mathcal{V}_\sigma$. Note that

$$|\mathcal{B}_\sigma| = \dim(\mathcal{V}_\sigma) = 2^{\omega(m/C(\sigma))},$$

where ω counts the number of prime divisors. If $C(\sigma) = m$, then $\mathcal{B}_\sigma$ is a singleton set consisting of a newform in $S_{k-\frac{1}{2}}^+(\Gamma_0(4m))$; if $C(\sigma) < m$ then all elements of $\mathcal{B}_\sigma$ are oldforms.

Now, define the set

$$(232) \quad \mathcal{B}_m = \bigcup_{\sigma} \mathcal{B}_\sigma,$$

where σ ranges over the set of all the cuspidal automorphic representations of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ that occur in $S_{k-\frac{1}{2}}^+(\Gamma_0(4m))$. Then $\mathcal{B}_m$ gives us an orthogonal basis of $S_{k-\frac{1}{2}}^+(\Gamma_0(4m))$. Given $h \in \mathcal{B}_m$, we let σ_h denote the automorphic representation of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ generated by the adelization Λ_h of h . Furthermore we define $h' \in S_{k-\frac{1}{2}}^+(\Gamma^0(4m), \chi_m)$ by

$$h'(\tau) = h\left(\frac{\tau}{4m}\right),$$

and we let $\Lambda_{h'}$ denote the adelization of h' . Note that $\Lambda_{h'}$ generates the representation $\sigma_h^{(m)}$ defined in (228).

For each $h \in \mathcal{B}_m$, the automorphic forms Λ_h and $\Lambda_{h'}$ are related by (226). The form $\Lambda_{h'}$ corresponds to a factorizable vector in $\sigma_h^{(m)}$, and at each prime dividing m its local component lies in $V_{m \cdot \sigma_{h,p}}(\Gamma^0(p\mathbb{Z}_p), \chi_{m,p})$. Since $\chi_{m,p}$ is real-valued, $\overline{\Lambda_{h'}}$ is a factorizable vector in $\overline{\sigma_h^{(m)}}$, with local component in

$$V_{m \cdot \overline{\sigma_{h,p}}}(\Gamma^0(p\mathbb{Z}_p), \chi_{m,p}).$$

By (229) and (230), the representation $\pi_{0,h}$ generated by the Shimura lift of h is

$$\pi_{0,h} = \mathrm{Wald}_\psi(\overline{\sigma_h}) = \mathrm{Wald}_{\psi^m}(\overline{\sigma_h^{(m)}}).$$

Note that $\pi_{0,h}$ is special at every prime dividing $C(\sigma_h)$, spherical at every prime not dividing $C(\sigma_h)$, and a holomorphic discrete series representation of weight $2k - 2$ at infinity.

3.6. Classical interpretation of Fourier–Jacobi periods. Let F be in $S_k(\mathrm{Sp}_4(\mathbb{Z}))$ with adelization Ψ_F . Let $\Lambda \in L^2(\mathrm{SL}_2(\mathbb{Q}) \backslash \widetilde{\mathrm{SL}}_2(\mathbb{A}))$ be an automorphic form on the metaplectic group, and let m be an odd, squarefree, positive integer. In the present data, the global Fourier–Jacobi period (7) is

$$(233) \quad \mathcal{FJ}_{\psi^m}(\Psi_F, \Lambda, \phi^{(m)}) := \int_{\mathrm{SL}_2(\mathbb{Q}) \backslash \mathrm{SL}_2(\mathbb{A})} \int_{H(\mathbb{Q}) \backslash H(\mathbb{A})} \Psi_F(ng) \Lambda(g) \overline{\Theta_{\psi^m}(ng, \phi^{(m)})} dn dg,$$

where we use the Tamagawa measure on $\mathrm{SL}_2(\mathbb{Q}) \backslash \mathrm{SL}_2(\mathbb{A})$ and the standard measure on $H(\mathbb{Q}) \backslash H(\mathbb{A})$. The main aim of this subsection is to give an interpretation of the global Fourier–Jacobi period defined by (233) in the classical language.

Recall that the Fourier–Jacobi coefficients (see (219)) of F have the usual theta decomposition

$$(234) \quad f_m(\tau, z) = \sum_{\alpha \pmod{2m}} f_{m,\alpha}(\tau) \theta_{m,\alpha}(\tau, z).$$

with $\theta_{m,\alpha}$ defined in (212). Set

$$(235) \quad f_m(\tau) = f_{m,F}(\tau) = \sum_{\alpha \pmod{2m}} f_{m,\alpha}(4m\tau).$$

Then we know [12, Thm. 5.6] that $f_m \in S_{k-\frac{1}{2}}^+(\Gamma_0(4m))$. We prove the following proposition.

Proposition 3.5. *Let m be an odd squarefree integer. Let $F \in S_k(\mathrm{Sp}_4(\mathbb{Z}))$ with adelization Ψ_F . Let $f_m \in S_{k-\frac{1}{2}}^+(\Gamma_0(4m))$ be given by (235). Let $h \in S_{k-\frac{1}{2}}^+(\Gamma_0(4m))$. Define*

$$h'(\tau) = h\left(\frac{\tau}{4m}\right),$$

so that $h' \in S_{k-\frac{1}{2}}^+(\Gamma^0(4m), \chi_m)$, and let Λ' be the adelization of h' . Then we have

$$\mathcal{FJ}_{\psi^m}(\Psi_F, \overline{\Lambda'}, \phi^{(m)}) = e^{-2\pi m} (4m)^{k-1} \frac{1}{2\xi(2)} \langle f_m, h \rangle.$$

Proof. Using the right invariance of the functions within the integral of (233), we can replace the integral over $H(\mathbb{Q}) \backslash H(\mathbb{A})$ by an integral over

$$n = \begin{bmatrix} 1 & \mu \\ \lambda & 1 \end{bmatrix} \begin{bmatrix} \mu & \kappa \\ 1 & -\lambda \end{bmatrix}, \quad \lambda, \mu, \kappa \in \mathbb{R}/\mathbb{Z},$$

and replace the integral over $\mathrm{SL}_2(\mathbb{Q}) \backslash \mathrm{SL}_2(\mathbb{A})$ by an integral over

$$g = \begin{bmatrix} \sqrt{y} & x/\sqrt{y} \\ & 1 \\ & 1/\sqrt{y} \\ & & 1 \end{bmatrix}, \quad x + iy \in \Gamma^0(4m) \backslash \mathbb{H}.$$

Put

$$\tau = x + iy, \quad z = \lambda\tau + \mu = \lambda x + \mu + i\lambda y, \quad \tau' = (\lambda^2 x + \lambda\mu + \kappa) + i(\lambda^2 y + 1),$$

so that in the integrals above, the variable τ ranges over $\Gamma^0(4m) \backslash \mathbb{H}$ and the variable z ranges over $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$. From the definitions (see (223)) we get

$$\begin{aligned} \Psi_F(ng) &= y^{k/2} F\left(\begin{bmatrix} \tau & z \\ z & \tau' \end{bmatrix}\right), \\ \overline{\Lambda'}(g) &= y^{\frac{k}{2}-\frac{1}{4}} \overline{h'(\tau)}. \end{aligned}$$

Using Proposition 3.1, we have

$$\Theta_{\psi^m}(ng, \phi^{(m)}) = y^{\frac{1}{4}} e^{2\pi i m(\kappa + \lambda^2 x + \lambda\mu)} e^{-2\pi m \lambda^2 y} \theta_m(\tau, z).$$

Therefore, using the fact that the Tamagawa measure on $\mathrm{SL}_2(\mathbb{A})$ is $(2\xi(2))^{-1}$ times the usual measure, we obtain

$$\begin{aligned} & 2\xi(2)\mathcal{FJ}_{\psi^m}(\Psi_F, \overline{\Lambda'}, \phi^{(m)}) \\ &= [\mathrm{SL}_2(\mathbb{Z}) : \Gamma^0(4m)]^{-1} \int_{\Gamma^0(4m) \backslash \mathbb{H}} \int_0^1 \int_0^1 \int_{\mathbb{R}/\mathbb{Z}} F\left(\begin{bmatrix} \tau & z \\ z & \tau' \end{bmatrix}\right) \overline{h'(\tau)\theta_m(\tau, z)} y^k \\ & \quad \times e^{-2\pi i m(\kappa + \lambda^2 x + \lambda \mu)} e^{-2\pi m \lambda^2 y} d\kappa \frac{d\lambda d\mu dx dy}{y^2}. \end{aligned}$$

Now inserting the Fourier–Jacobi expansion (219) we get

$$\begin{aligned} 2\xi(2)\mathcal{FJ}_{\psi^m}(\Psi_F, \overline{\Lambda'}, \phi^{(m)}) &= [\mathrm{SL}_2(\mathbb{Z}) : \Gamma^0(4m)]^{-1} \int_{\Gamma^0(4m) \backslash \mathbb{H}} \int_0^1 \int_0^1 \sum_{r=1}^{\infty} f_r(\tau, z) \overline{h'(\tau)\theta_m(\tau, z)} y^k \\ & \quad \times e^{2\pi i(r-m)(\lambda^2 x + \lambda \mu)} e^{-2\pi((m+r)\lambda^2 y + r)} \left(\int_{\mathbb{R}/\mathbb{Z}} e^{2\pi i(r-m)\kappa} d\kappa \right) \frac{d\lambda d\mu dx dy}{y^2} \\ &= [\mathrm{SL}_2(\mathbb{Z}) : \Gamma^0(4m)]^{-1} e^{-2\pi m} \int_{\Gamma^0(4m) \backslash \mathbb{H}} \int_0^1 \int_0^1 f_m(\tau, z) \overline{h'(\tau)\theta_m(\tau, z)} y^k \\ & \quad \times e^{-4\pi m \lambda^2 y} \frac{d\lambda d\mu dx dy}{y^2} \\ &= [\mathrm{SL}_2(\mathbb{Z}) : \Gamma^0(4m)]^{-1} e^{-2\pi m} \int_{\Gamma^0(4m) \backslash \mathbb{H}} \int_{\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}} f_m(\tau, z) \overline{h'(\tau)\theta_m(\tau, z)} y^k \\ & \quad \times e^{-4\pi m(y')^2/y} \frac{dx' dy' dx dy}{y^3}, \end{aligned}$$

where we change variables to write $z = x' + iy'$. Now, substituting the expression (211) for $\theta_m(\tau, z)$, the expression (234) for $f_m(\tau, z)$ and using the formula

$$\int_{z=x'+iy' \in \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}} \theta_{m,i}(\tau, z) \overline{\theta_{m,j}(\tau, z)} e^{-4\pi m(y')^2/y} dx' dy' = \sqrt{\frac{y}{4m}} \delta_{i,j}$$

(see [12, p. 60–61]), we get

$$2\xi(2)\mathcal{FJ}_{\psi^m}(\Psi_F, \overline{\Lambda'}, \phi^{(m)}) = \frac{e^{-2\pi m}}{2\sqrt{m}} \left\langle \sum_{\alpha \pmod{2m}} f_{m,\alpha}, h' \right\rangle = e^{-2\pi m} (4m)^{k-1} \langle f_m, h \rangle,$$

as required. $\square$

3.7. Archimedean calculations. Let π_∞ be the holomorphic discrete series representation of $\mathrm{Sp}_4(\mathbb{R})$ of scalar weight k , and let σ_∞ be the holomorphic discrete series representation of $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ of weight $k - \frac{1}{2}$. Let Ψ_∞ (resp., Λ_∞) be the lowest weight vector in π_∞ (resp., σ_∞). Let $\phi_\infty^{(m)}$ be as in (209). In this subsection, we evaluate the integral

$$\begin{aligned} & \alpha_\infty(\Psi_\infty, \overline{\Lambda_\infty}, \phi_\infty^{(m)}; \psi_\infty^m) \\ (236) \quad &= \int_{\mathrm{SL}_2(\mathbb{R})} \int_{H(\mathbb{R})} \langle \pi_\infty(n g) \Psi_\infty, \Psi_\infty \rangle \overline{\langle \sigma_\infty(g) \Lambda_\infty, \Lambda_\infty \rangle} \overline{\langle \omega_{\psi_\infty^m}(n g) \phi_\infty^{(m)}, \phi_\infty^{(m)} \rangle} dn dg. \end{aligned}$$

Here $\overline{\Lambda_\infty}$ is a vector in the antiholomorphic contragredient representation $\overline{\sigma_\infty}$.

Lemma 3.6. *With the above notations,*

$$(237) \quad \frac{\alpha_\infty(\Psi_\infty, \overline{\Lambda_\infty}, \phi_\infty^{(m)}; \psi_\infty^m)}{\langle \Psi_\infty, \Psi_\infty \rangle \langle \Lambda_\infty, \Lambda_\infty \rangle \langle \phi_\infty^{(m)}, \phi_\infty^{(m)} \rangle} = \frac{(4\pi)^{k+1} e^{-4\pi m} m^{k-2}}{(2k-3)\Gamma(k)}.$$

Proof. Writing

$$n = \begin{bmatrix} 1 & & b \\ a & 1 & c \\ & 1 & -a \end{bmatrix}, \quad g = \begin{bmatrix} e^t & & \\ & 1 & \\ & & e^{-t} \\ & & & 1 \end{bmatrix},$$

we can write the integral (236) as (see p. 132 of [40])

$$4\pi \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \langle \pi_\infty(ng)\Psi_\infty, \Psi_\infty \rangle \overline{\langle \sigma_\infty(g)\Lambda_\infty, \Lambda_\infty \rangle} \overline{\langle \omega_{\psi_\infty^m}(ng)\phi_\infty^{(m)}, \phi_\infty^{(m)} \rangle} \sinh(2t) \, dc \, da \, db \, dt.$$

By Lemmas 5.1, 5.2 and 5.3 of [40], we have

$$\begin{aligned} \frac{\langle \pi_\infty(ng)\Psi_\infty, \Psi_\infty \rangle}{\langle \Psi_\infty, \Psi_\infty \rangle} &= \left(\frac{4e^t}{a^2e^{2t} + b^2 + 2 + 2e^{2t} + i(ab(e^{2t} - 1) - c(e^{2t} + 1))} \right)^k, \\ \frac{\langle \sigma_\infty(g)\Lambda_\infty, \Lambda_\infty \rangle}{\langle \Lambda_\infty, \Lambda_\infty \rangle} &= \cosh(t)^{-k+\frac{1}{2}}, \\ \frac{\langle \omega_{\psi_\infty^m}(ng)\phi_\infty^{(m)}, \phi_\infty^{(m)} \rangle}{\langle \phi_\infty^{(m)}, \phi_\infty^{(m)} \rangle} &= \frac{e^{\frac{t}{2} - 2\pi m \frac{e^{2t}a^2 + b^2}{1+e^{2t}} - 2\pi im(ab \frac{e^{2t}-1}{e^{2t}+1} - c)}}{((1+e^{2t})/2)^{1/2}}. \end{aligned}$$

Note that, in the statement of Lemma 5.3 of [40], there is a typo and the denominator is missing. We will first do the c -integral. We have

$$\begin{aligned} &\int_{-\infty}^\infty \langle \pi_\infty(ng)\Psi_\infty, \Psi_\infty \rangle e^{-2\pi imc} \, dc \\ &= \int_{-\infty}^\infty \left(\frac{4e^t}{a^2e^{2t} + b^2 + 2 + 2e^{2t} + i(ab(e^{2t} - 1) - c(e^{2t} + 1))} \right)^k e^{-2\pi imc} \, dc \\ &= \left(\frac{m}{1+e^{2t}} \right)^k \int_{-\infty}^\infty \left(\frac{4e^t}{m \frac{a^2e^{2t} + b^2 + 2 + 2e^{2t}}{1+e^{2t}} + i(mab \frac{e^{2t}-1}{1+e^{2t}} - mc)} \right)^k e^{-2\pi imc} \, dc \\ &= \frac{m^{k-1}}{(1+e^{2t})^k} \int_{-\infty}^\infty \left(\frac{4e^t}{m \frac{a^2e^{2t} + b^2 + 2 + 2e^{2t}}{1+e^{2t}} + i(mab \frac{e^{2t}-1}{1+e^{2t}} - c)} \right)^k e^{-2\pi ic} \, dc \\ &= \frac{(8\pi e^t)^k}{\Gamma(k)} \frac{m^{k-1}}{(1+e^{2t})^k} e^{-2\pi m \frac{a^2e^{2t} + b^2 + 2 + 2e^{2t}}{1+e^{2t}} - 2\pi imab \frac{e^{2t}-1}{1+e^{2t}}}. \end{aligned}$$

In the last integral, we have used Formula 3.382 (7) of Gradshteyn and Ryzhik [16]. Now the inner integral over a and b is given by

$$\begin{aligned} &\frac{(8\pi e^t)^k}{\Gamma(k)} \frac{m^{k-1}}{(1+e^{2t})^k} \frac{e^{\frac{t}{2}}}{2^{-1/2}(1+e^{2t})^{1/2}} \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-4\pi m \frac{a^2e^{2t} + b^2 + 1 + e^{2t}}{1+e^{2t}}} \, da \, db \\ &= \frac{2^{2k-1} \pi^k e^{-4\pi m} m^{k-2}}{\Gamma(k)} \cosh(t)^{-k+\frac{1}{2}}. \end{aligned}$$

Here, we have used the standard formula $\int_{-\infty}^\infty e^{-x^2} \, dx = \sqrt{\pi}$, and a suitable change of variables. (Note that in [40], the author has $4^{k+1/2}$ instead of 4^k .) The remaining t -integral is elementary, giving the formula (237). $\square$

3.8. Main results. Recall that, for each irreducible automorphic representation σ of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ occurring in $S_{k-\frac{1}{2}}^+(\Gamma_0(4m))$, we constructed in Section 3.5 an orthogonal basis $\mathcal{B}_\sigma \subset S_{k-\frac{1}{2}}^+(\Gamma_0(4m))$ of the σ -isotypic subspace $\mathcal{V}_\sigma$. We now prove our main theorem.

Theorem 3.7. *Let k be a positive, even integer, let m be a positive, odd, squarefree integer, and let $F \in S_k(\mathrm{Sp}_4(\mathbb{Z}))$ be a Hecke eigenform that is not a Saito–Kurokawa lift.⁷ Let $f_m \in S_{k-\frac{1}{2}}^+(\Gamma_0(4m))$ be given by (235). Let σ be an irreducible automorphic representation of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ occurring in $S_{k-\frac{1}{2}}^+(\Gamma_0(4m))$ and put $C = C(\sigma)$, so that $C \mid m$. Let π_0 be the irreducible automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ occurring in $S_{2k-2}(\Gamma_0(m))$ that is associated to σ via the classical Shimura–Waldspurger correspondence; precisely, by (229),*

$$\pi_0 = \mathrm{Wald}_\psi(\bar{\sigma}).$$

The arithmetic conductor satisfies $C(\pi_0) = C(\bar{\sigma}) = C(\sigma) = C$.

For each prime p , let $\alpha_p, \alpha_p^{-1}, \beta_p, \beta_p^{-1}$ be the Satake parameters of π_F at p . For each prime $p \nmid C$, let δ_p, δ_p^{-1} be the Satake parameters of π_0 at p . For each prime $p \mid C$, let w_p be the local Atkin–Lehner eigenvalue of π_0 at p .

Assume the truth of Conjecture 1. Then

$$\sum_{h \in \mathcal{B}_\sigma} \frac{|\langle f_m, h \rangle|^2}{\langle h, h \rangle} = \langle F, F \rangle \frac{4\pi^{k+5}}{3(2k-3)\Gamma(k)} \cdot \frac{L(1/2, \pi_F \times \pi_0)}{L(1, \pi_F, \mathrm{Ad})L(1, \pi_0, \mathrm{Ad})} \prod_{p \mid m} r_p(F, \pi_0).$$

Above, the global L -functions do not include the archimedean L -factors, and the quantities $r_p(F, \pi_0)$ are given by

$$r_p(F, \pi_0) = \begin{cases} \frac{2}{p+1} \cdot \frac{(\alpha_p^{-1/2} + \alpha_p^{1/2})^2 (\beta_p^{-1/2} + \beta_p^{1/2})^2}{(1 + \delta_p p^{-1/2})(1 + \delta_p^{-1} p^{-1/2})} & \text{if } p \nmid C, \\ \frac{2}{p+1} & \text{if } p \mid C \text{ and } w_p = 1, \\ 0 & \text{if } p \mid C \text{ and } w_p = -1. \end{cases}$$

Proof. Using Proposition 3.5, we see that for each $h \in \mathcal{B}_\sigma$ we have

$$(238) \quad |\langle f_m, h \rangle|^2 = e^{4\pi m} (4m)^{2-2k} 4\xi(2)^2 |\mathcal{FJ}_{\psi^m}(\Psi_F, \overline{\Lambda_{h'}}, \phi^{(m)})|^2.$$

For each $h \in \mathcal{B}_\sigma$, let $\Lambda_{h'}$ be the adelization of h' , as in Section 3.5. The form $\Lambda_{h'}$ generates $\frac{\sigma_h^{(m)}}{\sigma_h^{(m)}}$, whereas the form that occurs in (238), namely $\overline{\Lambda_{h'}}$, generates the contragredient representation $\frac{\sigma_h^{(m)}}{\sigma_h^{(m)}}$. Complex conjugation preserves the Petersson norm, so (210), (217), and (227) give

$$(239) \quad \frac{\langle \Psi_F, \Psi_F \rangle \langle \Lambda_{h'}, \Lambda_{h'} \rangle \langle \phi^{(m)}, \phi^{(m)} \rangle}{\langle F, F \rangle \langle h, h \rangle} = \frac{1}{8\xi^2(2)\xi(4)} (4m)^k.$$

Now assume the truth of Conjecture 1, applied to the representation $\sigma = \overline{\sigma_h^{(m)}}$ that contains the vector $\Lambda = \overline{\Lambda_{h'}}$. By (230),

$$L_{\psi^m}(s, \pi_F \times \overline{\sigma_h^{(m)}}) = L(s, \pi_F \times \pi_0), \quad L_{\psi^m}(s, \overline{\sigma_h^{(m)}}, \mathrm{Ad}) = L(s, \pi_0, \mathrm{Ad}).$$

The constant C_G in (8) equals $\xi(2)^{-1}$. The local integral $\alpha_p^\#(\Psi_p, \overline{\Lambda_{h',p}}, \phi_p^{(m)}; \psi_p^m)$ is equal to 1 at all finite places $v = p$ that do not divide $2m$ by [39]. At $v = 2$, the calculation of [40, Proposition 4.3 and Section 6] is already made with the conjugated metaplectic vector and gives

$$\alpha_2^\#(\Psi_2, \overline{\Lambda_{h',2}}, \phi_2^{(m)}; \psi_2^m) = \frac{1}{2}.$$

⁷The smallest such k is 20.

At $v = \infty$, we have by (237), $\frac{\alpha_\infty(\Psi_\infty, \overline{\Lambda_{h',\infty}}, \phi_\infty^{(m)}; \psi_\infty^m)}{\langle \Psi_\infty, \Psi_\infty \rangle \langle \Lambda_{h',\infty}, \Lambda_{h',\infty} \rangle \langle \phi_\infty^{(m)}, \phi_\infty^{(m)} \rangle} = \frac{2^6 \pi^{k+1} e^{-4m\pi} (4m)^{k-2}}{(2k-3)\Gamma(k)}$. The parameter β in (8) equals 1 since F has full level and is not a Saito–Kurokawa lift, and so the degree-five standard transfer of π_F to $\mathrm{GL}_5(\mathbb{A})$ is cuspidal. So (8) becomes

$$(240) \quad \frac{|\mathcal{FJ}_{\psi^m}(\Psi_F, \overline{\Lambda_{h'}}, \phi^{(m)})|^2}{\langle \Psi_F, \Psi_F \rangle \langle \Lambda_{h'}, \Lambda_{h'} \rangle \langle \phi^{(m)}, \phi^{(m)} \rangle} = \zeta(2)\zeta(4) \frac{2^4 \pi^{k+1} e^{-4m\pi} (4m)^{k-2}}{(2k-3)\Gamma(k)} \cdot \frac{L(1/2, \pi_F \times \pi_0)}{L(1, \pi_F, \mathrm{Ad})L(1, \pi_0, \mathrm{Ad})} \times \prod_{p|m} \alpha_p^\#(\Psi_p, \overline{\Lambda_{h',p}}, \phi_p^{(m)}; \psi_p^m).$$

Multiplying together (238), (239) and (240) gives

$$(241) \quad \frac{|\langle f_m, h \rangle|^2}{\langle h, h \rangle} = \langle F, F \rangle \frac{4\pi^{k+5}}{3(2k-3)\Gamma(k)} \cdot \frac{L(1/2, \pi_F \times \pi_0)}{L(1, \pi_F, \mathrm{Ad})L(1, \pi_0, \mathrm{Ad})} \times \prod_{p|m} \alpha_p^\#(\Psi_p, \overline{\Lambda_{h',p}}, \phi_p^{(m)}; \psi_p^m).$$

Therefore, observing (231),

$$(242) \quad \begin{aligned} \sum_{h \in \mathcal{B}_\sigma} \frac{|\langle f_m, h \rangle|^2}{\langle h, h \rangle} &= \sum_{i \mid \frac{m}{C(\sigma)}} \frac{|\langle f_m, h_{\sigma,i} \rangle|^2}{\langle h_{\sigma,i}, h_{\sigma,i} \rangle} \\ &= \langle F, F \rangle \frac{4\pi^{k+5}}{3(2k-3)\Gamma(k)} \frac{L(1/2, \pi_F \times \pi_0)}{L(1, \pi_F, \mathrm{Ad})L(1, \pi_0, \mathrm{Ad})} \times \sum_{i \mid \frac{m}{C(\sigma)}} \prod_{p|m} \alpha_p^\#(\Psi_p, \overline{\Lambda_{h',i,p}}, \phi_p^{(m)}; \psi_p^m) \\ &= \langle F, F \rangle \frac{4\pi^{k+5}}{3(2k-3)\Gamma(k)} \frac{L(1/2, \pi_F \times \pi_0)}{L(1, \pi_F, \mathrm{Ad})L(1, \pi_0, \mathrm{Ad})} \times \prod_{p|m} \alpha_p^\#(\pi_{F,p}, \overline{\sigma_p}; m), \end{aligned}$$

The last factor above, for each $p|m$, is exactly the quantity defined in (32) with the representation $\overline{\sigma_p}$. The result follows from Theorem 2.2. $\square$

Corollary 3.8. *Keep the notations and assumptions of Theorem 3.7, and define the quantity $r_m(F, \pi_0) := \prod_{p|m} r_p(F, \pi_0)$. Then*

$$\frac{\langle f_m, f_m \rangle}{\langle F, F \rangle} = \frac{4\pi^{k+5}}{3(2k-3)\Gamma(k)} \sum_{\pi_0} r_m(F, \pi_0) \frac{L(1/2, \pi_F \times \pi_0)}{L(1, \pi_F, \mathrm{Ad})L(1, \pi_0, \mathrm{Ad})},$$

where π_0 traverses the set of irreducible cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A})$ that occur in $S_{2k-2}(\Gamma_0(m))$.

Proof. Using orthogonality, with $\mathcal{B}_m$ as in (232) we have

$$(243) \quad \langle f_m, f_m \rangle = \sum_{h \in \mathcal{B}_m} \frac{|\langle f_m, h \rangle|^2}{\langle h, h \rangle} = \sum_{\sigma} \sum_{h \in \mathcal{B}_\sigma} \frac{|\langle f_m, h \rangle|^2}{\langle h, h \rangle}.$$

By the Shimura–Waldspurger correspondence for the Kohnen plus space of squarefree level, the map $\sigma \mapsto \mathrm{Wald}_\psi(\overline{\sigma}) = \pi_0$ is a bijection between the irreducible genuine automorphic representations occurring in $S_{k-\frac{1}{2}}^+(\Gamma_0(4m))$ and the irreducible cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A})$ occurring in $S_{2k-2}(\Gamma_0(m))$. Hence the outer sum over σ in (243) may be reindexed as a sum over π_0 , and the result follows from Theorem 3.7. $\square$

3.9. Upper bounds on Petersson inner products and Fourier coefficients. In this subsection, we show how our main result, together with the Generalized Riemann Hypothesis (GRH), implies strong upper bounds on Petersson inner products of half-integral weight forms and Fourier coefficients of Siegel cusp forms.

Proposition 3.9. *Let k be a positive even integer, let m be a positive odd squarefree integer, and let $F \in S_k(\mathrm{Sp}_4(\mathbb{Z}))$ be a Hecke eigenform that is not a Saito–Kurokawa lift. Let $f_m \in S_{k-\frac{1}{2}}^+(\Gamma_0(4m))$ be given by (235). Let $h \in S_{k-\frac{1}{2}}^+(\Gamma_0(4m))$ be an eigenform for the Hecke operators $T(n^2)$ for all $(n, 4m) = 1$. Assume Conjecture 1 and also assume GRH for the L -functions appearing in Theorem 3.7. Then*

$$|\langle f_m, h \rangle| \ll_\epsilon m^{-1/2} (km)^\epsilon \langle F, F \rangle^{1/2} \langle h, h \rangle^{1/2} \frac{\pi^{k/2}}{k^{1/2} \Gamma(k)^{1/2}}.$$

Proof. Since h is an eigenform for the Hecke operators $T(n^2)$ for all $(n, 4m) = 1$, there is a genuine, irreducible, cuspidal automorphic representation σ of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ such that $h \in \mathcal{V}_\sigma$. Recall that $\dim \mathcal{V}_\sigma \ll_\epsilon m^\epsilon$. Thus, after changing ϵ , it would suffice to prove the result for $h \in \mathcal{B}_\sigma$, and we assume this henceforth.

By Theorem 3.7,

$$\frac{|\langle f_m, h \rangle|^2}{\langle h, h \rangle} \leq \langle F, F \rangle \frac{4\pi^{k+5}}{3(2k-3)\Gamma(k)} \frac{L(1/2, \pi_F \times \pi_0)}{L(1, \pi_F, \mathrm{Ad})L(1, \pi_0, \mathrm{Ad})} \prod_{p|m} r_p(F, \pi_0).$$

Under GRH, we have

$$\frac{L(1/2, \pi_F \times \pi_0)}{L(1, \pi_F, \mathrm{Ad})L(1, \pi_0, \mathrm{Ad})} \ll_\epsilon (km)^\epsilon.$$

We next bound the local factors. By temperedness, the Satake parameters $\alpha_p, \beta_p, \delta_p$ have absolute value one. Thus, if $p \mid m$, then $r_p(F, \pi_0) \ll \frac{1}{p}$ and consequently,

$$\prod_{p|m} r_p(F, \pi_0) \ll_\epsilon m^{-1+\epsilon}.$$

Combining these estimates with Theorem 3.7, and using $\frac{4\pi^{k+5}}{3(2k-3)\Gamma(k)} \ll \frac{\pi^k}{k\Gamma(k)}$, we obtain the desired result (after replacing ϵ by 2ϵ). $\square$

Next, recall that the n th Fourier coefficient of f_m is equal to $\sum_{\substack{0 \leq r \leq 2m-1 \\ r^2 \equiv -n \pmod{4m}}} a\left(F, \begin{bmatrix} \frac{n+r^2}{4m} & \frac{r}{2} \\ \frac{r}{2} & m \end{bmatrix}\right)$. If $m = 1$, the sum above consists of only one term. If $m = p$ is a prime, then the sum consists of 0, 1, or 2 terms, depending on whether $\left(\frac{-n}{p}\right)$ equals -1, 0 or 1. But in the case there are two terms, these two Fourier coefficients of F are equal because the corresponding symmetric matrices are equivalent under the action of $\mathrm{GL}_2(\mathbb{Z})$ (note that k is even). By combining this observation with the previous proposition we obtain the following bound on the Fourier coefficients of F .

Proposition 3.10. *Let k be a positive even integer, and let $F \in S_k(\mathrm{Sp}_4(\mathbb{Z}))$ be a Hecke eigenform that is not a Saito–Kurokawa lift. Assume Conjecture 1 and GRH. Then, for all $S = \begin{bmatrix} a & b/2 \\ b/2 & m \end{bmatrix}$ with m equal to 1 or equal to an odd prime, we have*

$$\frac{|a(F, S)|}{\langle F, F \rangle^{1/2}} \ll_\epsilon (km)^\epsilon m^{1/2} k^{3/4} \frac{(4\pi)^k}{\Gamma(k)} (\det(S))^{\frac{k}{2} - \frac{3}{4} + \epsilon}.$$

Proof. Write

$$f_m(z) = \sum_{n \geq 1} a_m(n) e^{2\pi i n z}$$

and put $n = 4 \det(S)$. By the observation preceding the proposition,

$$|a(F, S)| \leq |a_m(n)|.$$

Expanding f_m in the orthogonal basis $\mathcal{B}_m$, we obtain

$$|a_m(n)| \leq \sum_{h \in \mathcal{B}_m} \frac{|\langle f_m, h \rangle|}{\langle h, h \rangle^{1/2}} \frac{|a_h(n)|}{\langle h, h \rangle^{1/2}}.$$

By the preceding proposition,

$$\frac{|\langle f_m, h \rangle|}{\langle h, h \rangle^{1/2}} \ll_{\epsilon} m^{-1/2} (km)^{\epsilon} \langle F, F \rangle^{1/2} \frac{\pi^{k/2}}{k^{1/2} \Gamma(k)^{1/2}}.$$

On the other hand, Waldspurger's theorem [36] in the explicit form of [5, Theorem 10.1] (see also [19, Proposition 2.2]), together with GRH and the standard Hecke relations for Fourier coefficients whose indices differ by a square (see [19, Lemma 2.1]) give

$$\sum_{h \in \mathcal{B}_m} \frac{|a_h(n)|}{\langle h, h \rangle^{1/2}} \ll_{\epsilon} (kmn)^{\epsilon} km \frac{2^{k-1} \pi^{k/2+1/4}}{\Gamma(k - \frac{1}{2})^{1/2}} n^{\frac{k}{2} - \frac{3}{4}}.$$

Here we have also used the usual dimension estimate

$$\dim S_{k-\frac{1}{2}}^+(\Gamma_0(4m)) \ll km.$$

Combining the preceding estimates gives

$$\frac{|a(F, S)|}{\langle F, F \rangle^{1/2}} \ll_{\epsilon} (kmn)^{\epsilon} m^{1/2} \frac{2^{k-1} \pi^{k+1/4} k^{1/2}}{\Gamma(k)^{1/2} \Gamma(k - \frac{1}{2})^{1/2}} n^{\frac{k}{2} - \frac{3}{4}}.$$

Since $n = 4 \det(S)$, the duplication formula and Stirling's formula give

$$\frac{2^{k-1} \pi^{k+1/4} k^{1/2}}{\Gamma(k)^{1/2} \Gamma(k - \frac{1}{2})^{1/2}} 4^{\frac{k}{2} - \frac{3}{4}} \ll \frac{k^{3/4} (4\pi)^k}{\Gamma(k)}.$$

This completes the proof. $\square$

Now suppose that S is primitive. Then we know that it represents infinitely many primes (in fact, it represents a positive proportion of primes; see [18, Theorem 1 (i)]). Put $\min_{\text{pr}} S = 1$ if S represents 1, and let $\min_{\text{pr}} S$ be the smallest odd prime represented by S otherwise, so that⁸ $\min_{\text{pr}} S < \infty$. We can replace S by a matrix S' in its $\text{SL}_2(\mathbb{Z})$ -equivalence such that $a(F, S) = a(F, S')$ and the bottom right entry of S' equals $\min_{\text{pr}} S$. Therefore, Proposition 3.10 may be rephrased as

$$(244) \quad \frac{|a(F, S)|}{\langle F, F \rangle^{1/2}} \ll_{\epsilon} k^{\epsilon} (\min_{\text{pr}} S)^{1/2+\epsilon} k^{3/4} \frac{(4\pi)^k}{\Gamma(k)} (\det(S))^{\frac{k}{2} - \frac{3}{4} + \epsilon}.$$

The bound (244) may be viewed as conditional progress toward a deep conjecture of Resnikoff and Saldana [29], which predicts that

$$(245) \quad |a(F, S)| \ll_{F, \epsilon} |\det(S)|^{\frac{k}{2} - \frac{3}{4} + \epsilon}.$$

It is interesting to compare the conditional bound (244) with the conditional bound obtained in [10, Corollary 1.2], where it was shown, assuming GRH, that

$$\frac{|a(F, S)|}{\langle F, F \rangle^{1/2}} \ll_{\epsilon} k^{\epsilon} k^{1/2} \frac{(4\pi)^k}{\Gamma(k)} \det(S)^{\frac{k-1}{2} + \epsilon}.$$

The bound in [10] relied on the GGP identity for Bessel periods, in contrast to the bound (244), which uses the GGP identity for Fourier–Jacobi periods.

The best currently known unconditional bound toward (245) is due to Kohnen [23], who proved that $|a(F, S)| \ll_{F, \epsilon} |\det(S)|^{\frac{k}{2} - \frac{13}{36} + \epsilon}$. Kohnen's result relies on the following bound:

$$|a(F, S)| \ll_{F, \epsilon} (\min S)^{5/18+\epsilon} \det(S)^{\frac{k}{2} - \frac{1}{2} + \epsilon},$$

⁸Under GRH, it can be shown that for “typical” S , we have $\min_{\text{pr}} S \ll_{\epsilon} (\det S)^{1/2+\epsilon}$; see [33, Corollary 1.4].

where $\min S$ denotes the smallest integer represented by S . We also refer to a recent paper of Assing [4] for some related results.

3.10. Non-vanishing. We end with an application of our main result to non-vanishing of central L -values.

Proposition 3.11. *Let k be a positive even integer, and let $F \in S_k(\mathrm{Sp}_4(\mathbb{Z}))$ be a Hecke eigenform that is not a Saito–Kurokawa lift. Assume Conjecture 1. Then, for each positive, odd, squarefree integer m such that the form $f_m \in S_{k-\frac{1}{2}}^+(\Gamma_0(4m))$ defined in (235) is nonzero, there exists a classical newform g of weight $2k-2$, trivial character, and level dividing m such that*

$$L\left(\frac{1}{2}, \pi_F \times \pi_g\right) \neq 0.$$

Proof. This is immediate from Corollary 3.8. □

Remark 3.12. *It is known (see, e.g., Section 5.2 of [19]) that the set of primes p such that $f_p \neq 0$ has positive density. Proposition 3.11 implies that for each such prime p , there is a newform $g \in S_{2k-2}(\Gamma_0(m))$ with $m \in \{1, p\}$ with $L(1/2, \pi_F \times \pi_g) \neq 0$.*

Remark 3.13. *A recent theorem of Manickam [24, Theorem 1] asserts that the first Fourier–Jacobi coefficient of F is nonzero. This claim, if true, would imply, by [12, Theorem 5.4] and the previous proposition, that $L(\frac{1}{2}, \pi_F \times \pi_g) \neq 0$ for some Hecke eigenform $g \in S_{2k-2}(\mathrm{SL}_2(\mathbb{Z}))$. This would be a remarkable nonvanishing result: the family has size $\asymp k$, while the analytic conductor is $\asymp k^6$.*

Unfortunately, there appears to be a gap on page 410 of [24]. The identity

$$(246) \quad C_\psi(D, r) = a_{\psi|Z_p}(|D|)$$

given there is incompatible with the Eichler–Zagier map Z_p (defined correctly on page 406). Indeed, if $D = r^2 - 4np < 0$ and $(r, p) = 1$, then the two square roots of D modulo $2p$ are r and $-r$, and hence the correct identity (using that k is even) is

$$(247) \quad 2C_\psi(D, r) = a_{\psi|Z_p}(|D|).$$

Once this correction is made, the two equations compared at the bottom of page 410 are identical, and their subtraction yields only $0 = 0$. Hence the claimed contradiction, and therefore the proof of [24, Theorem 1], does not follow.⁹

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⁹When we communicated this issue to Manickam, he responded that the map used in his proof is a normalized Eichler–Zagier isomorphism on the newspace, obtained in his earlier work with Ramakrishnan [25]. However, this does not appear to resolve the issue; the normalization required to make (246) hold amounts to replacing Z_p by $\frac{1}{2}Z_p$, which also divides the coefficient corresponding to p^2D by 2. Thus the two equations being compared on page 410 remain identical. See also [2, Footnote 4], where it is noted that the factors R_D were omitted from the relevant theorem of Manickam–Ramakrishnan [25].

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