Analytic aspects of automorphic forms

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Key area of research: Understand better the **asymptotics** and **analytic properties** of these eigenfunctions.

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The Quantum Unique Ergodicity (QUE) Conjecture

Let X be a hyperbolic surface of finite volume and f_n traverse a sequence of eigenfunctions on X with $\langle f_n, f_n \rangle = 1$ and eigenvalues $\lambda_n \to \infty$. Then, for any compact subset C of X,

$$\lim_{n\to\infty} \int_C |f_n(z)|^2 d\mu(z) = \frac{\mathrm{vol}(C)}{\mathrm{vol}(X)}.$$

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Quantum mechanical interpretation: Eigenfunctions correspond to particles, eigenvalues correspond to their energies.

Number Theory enters the picture

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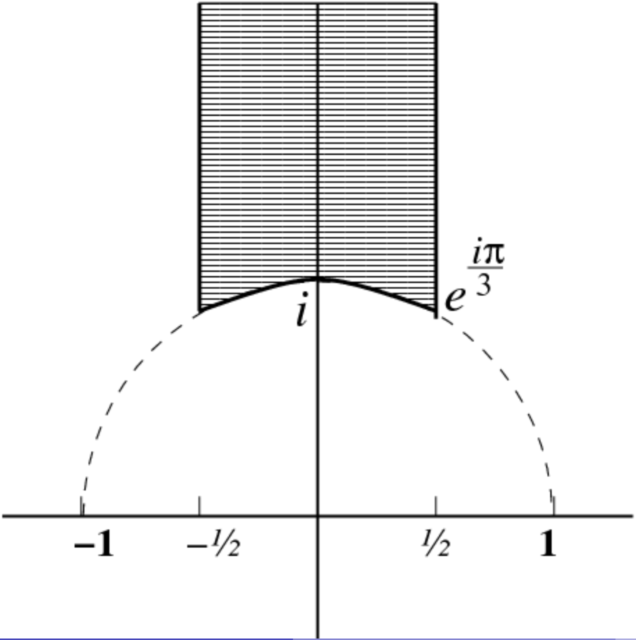
A protypical example of an arithmetic surface is $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$.
The Laplacian takes the form

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

A picture of $X = \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$



Arithmetic QUE

The celebrated arithmetic QUE proved by Lindenstrauss (2006) and Soundararajan (2010)

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- One of the reasons Lindenstrauss won the Fields medal.

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Iwaniec and Sarnak, 1995

Let $X = \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ be now arithmetic, f eigenfunction of Laplacian with $\langle f, f\rangle = 1$ and eigenvalue λ. Then,

$$\|f\|_\infty \ll_X \lambda^{5/24+\epsilon}.$$

The connection to number theory

When X is arithmetic, say $X = \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$,

- **Hecke operators:** There exist certain Hecke correspondences on X, leading to Hecke operators on the space of functions on X.

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When X is arithmetic, say $X = \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$,

- **Hecke operators:** There exist certain Hecke correspondences on X, leading to Hecke operators on the space of functions on X.
- **Automorphic forms:** Their Hecke-Laplace eigenfunctions are examples of automorphic forms, which can be defined in much greater generality, and come with L-functions and a rich theory (Langlands program, automorphic representations, deep conjectures)

What are Hecke operators?

- Let $X = \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$, p a prime. We define

$$(T_p(f))(z) = \sum_{\gamma \in \mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{Z})\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\mathrm{SL}_2(\mathbb{Z})} f(\gamma z).$$

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- In any case, it is natural to focus on joint Hecke-Laplace eigenfunctions, and the additional symmetries allow one to prove results are otherwise inaccessible (Lindenstrauss, Sarnak, ...)

The connection to automorphic representations and L-functions

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L-functions

We define the L-function attached to f

$$L(s, f) = \sum_{n>0} \frac{a_n}{n^{s+1/2}}.$$

It turns out that $L(s, f)$ extends to a holomorphic function on the entire complex plane and has a functional equation taking $s \mapsto 1 - s$.

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2. It allows us to generalize and extend these questions naturally into new directions.
3. It gives us powerful tools to solve these problems.

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All zeroes of $L(s, f \times f \times g)$ lie on the line $\mathrm{Re}(s) = 1/2$.

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And much weaker ...

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Subconvexity problem

Prove that $L(1/2, f \times f \times g) \ll_g \lambda^{1-\delta}$ for some $\delta > 0$.

Unfortunately this is still completely open.

Subconvexity: $L(1/2, f \times f \times g) \ll_g \lambda^{1-\delta}$.

QUE

Given Hecke-Laplace eigenfunctions f_n, g on $X = \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ that vanish at infinity, with $\lambda_n \to \infty$, we have

$$\lim_{n\to\infty} \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} |f_n(z)|^2 g(z) \frac{dxdy}{y^2} = 0.$$

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Watson's formula

In the above setup, we have

$$\left(\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} |f_n(z)|^2 g(z) \frac{dxdy}{y^2} \right)^2 = C(f_n, g)\lambda_n^{-1} L(1/2, f_n \times f_n \times g)$$

where $C(f_n, g)$ grows slower than any polynomial in λ_n.

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Conclusion: The subconvexity problem is essentially equivalent to QUE.

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So this gives us two natural extensions of the QUE and sup-norm problems.

1. Replace the condition of being a Laplace eigenfunction with being a holomorphic modular form (and λ by k). (The holomorphic analogue)
2. Replace $\mathrm{SL}_2(\mathbb{Z})$ by a suitable subgroup. (The level aspect)

The holomorphic modular forms

The Ramanujan Δ-function is defined on the upper-half plane $\mathbb{H}$ as follows:

$$\Delta(z) = e^{2\pi iz} \prod_{n=1}^{\infty} (1 - e^{2\pi inz})^{24} = \sum_{n=1}^{\infty} \tau(n) e^{2\pi inz}.$$

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Famous conjectures of Ramanujan:

- $\tau(mn) = \tau(m)\tau(n)$ if $(m, n) = 1$. Proved by **Mordell** (1917)
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$\Delta(z)$ is a holomorphic modular form of weight 12:

$$\Delta\left(\frac{az+b}{cz+d}\right) = (cz+d)^{12}\Delta(z) \text{ for } z \in \mathbb{H}, \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{Z}).$$

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- Neither approach gives the complete answer, but if one approach fails, it can be shown that the other succeeds!!
- Holowinsky + Soundararajan = QUE for holomorphic modular forms (Annals, 2010)

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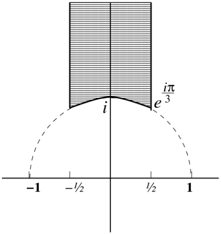
- From the point of view of automorphic representations, **varying** N is on exactly the same footing as **varying** λ. Corresponds respectively to the **non-archimedean** and **archimedean** primes.

Some pictures of $X_0(N) = \Gamma_0(N)\backslash\mathbb{H}$



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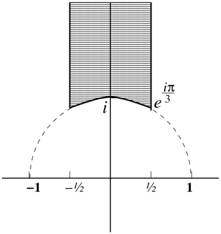

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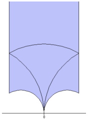
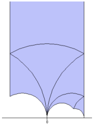
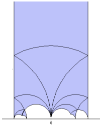


N=3,
genus=0,
cusps=2.

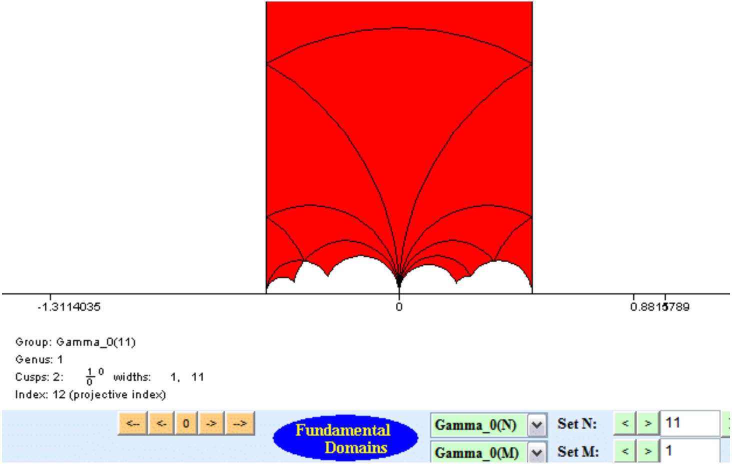


N=4,
genus=0,
cusps=3.



N=6,
genus=0,
cusps=6.

A picture of $X_0(11) = \Gamma_0(11)\backslash\mathbb{H}$ (credit: Verrill)



N=11, genus =1, cusps =2.

Level aspect QUE

Pitale-Nelson-Saha, published in JAMS in 2014

Let p be a fixed prime, and let f_n ($n \to \infty$) traverse a sequence of L^2-normalized Hecke-Laplace eigenfunctions on X_{p^n} whose eigenvalues stay bounded. Let $r_n : X_1 \to X_{p^n}$ be the natural map. Then, for any compact subset C of X_1,

$$\lim_{n \to \infty} \int_{r_n^{-1}(C)} |\phi_n(z)|^2 \frac{dxdy}{y^2} = \frac{\mathrm{vol}(C)}{\mathrm{vol}(X_1)}.$$

Actually proved holomorphic analogue.

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- Subconvexity in the level aspect: For f weight k on $\Gamma_0(N)$, we have $L(1/2, f \times f \times g) \ll_g (Nk)^{1-\delta}$ for some $\delta > 0$. Recently proved for $N =$ a prime, by Munshi-Nelson in a remarkable work.

What about the sup-norm problem in level aspect?

$\|f\|_\infty \ll$	Due to	Year	Restriction
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Very recent work (Hu-Nelson-Saha): Optimum bound of $N^{1/8}$ for *minimal* eigenfunctions.

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6. What about higher rank groups?

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- Circle of implications linking QUE, subconvexity, period formulas for L-functions, and the sup-norm problem, not fully understood.

Thank you!