The Manin constant, the modular degree, and Fourier expansions at cusps

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(Joint work with K Česnavičius and M Neururer)

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Some facts on cusps for $\Gamma_0(N)$

- Any cusp $\mathfrak{c} \in X_0(N)(\mathbb{C})$ is equivalent to

$$\mathfrak{c} = \frac{a}{L}, \text{ for some } L|N, \ \gcd(a, L) = 1.$$

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- The width of a cusp $\mathfrak{c} = \frac{a}{L}$ equals

$$w(\mathfrak{c}) = \frac{N}{\gcd(L^2,\ N)}.$$

$w(\mathfrak{c})$ is the smallest integer w such that $\left(\begin{smallmatrix} a & * \\ L & * \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & w \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & * \\ L & * \end{smallmatrix}\right)^{-1} \in \Gamma_0(N)$.

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- The Atkin-Lehner involutions: Let $\mathfrak{c} = \frac{a}{L}$ be a cusp. Then there exists an Atkin-Lehner involution taking $\mathfrak{c}$ to a cusp of denominator L' iff $\mathrm{val}_p(L') \in \{\mathrm{val}_p(L), \mathrm{val}_p(N) - \mathrm{val}_p(L)\}$ for each $p|N$.

The main question

- Let $f = \sum_{n>0} a_f(n)q^n, q = e^{2\pi iz}$ be a holomorphic newform of weight k, level N, trivial character.
- Normalize $a_f(1) = 1$. Then it well-known that all $a_f(n) \in \overline{\mathbb{Z}}$.

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- Fourier expansion at $\mathfrak{c}$: Let $\mathfrak{c} = \gamma\infty$ with $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

$$(f|_k\gamma)(z) = \sum_{n\geq 0} a_f(n;\mathfrak{c}) q^{\frac{n}{w(\mathfrak{c})}}.$$

  Note: $a_f(n;\mathfrak{c})$ only well-defined up to a $w(\mathfrak{c})$'th root of unity.
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- What can we say about the "denominators" of $a_f(n;\mathfrak{c})$?

For a prime p, we are interested in good *lower* bounds for

$$\mathrm{val}_p(f|_\mathfrak{c}) := \inf_{n\geq 0}(\mathrm{val}_p(a_f(n;\mathfrak{c}))).$$

Here, $\mathrm{val}_p\colon \overline{\mathbb{Q}}_p \to \mathbb{Q}\cup\{\infty\}$ is the p-adic valuation with $\mathrm{val}_p(p) = 1$, extended to $\overline{\mathbb{C}}$ via any fixed choice of isomorphism $\mathbb{C} \simeq \overline{\mathbb{Q}}_p$.

Let f be a normalized newform for $\Gamma_0(N)$ of weight k.

Find good *lower* bounds for $\mathrm{val}_p(f|_{\mathfrak{c}}) := \inf_{n\geq 0}(\mathrm{val}_p(a_f(n;\mathfrak{c})))$.

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1. Clearly, $\mathrm{val}_p(f|_\infty) = 0$.
2. The q-expansion principle: If the Fourier coefficients at infinity lie in a ring R, then the Fourier coefficients at any cusp lie in $R[1/N, e^{\frac{2\pi i}{N}}]$. In particular, $\mathrm{val}_p(f|_{\mathfrak{c}}) = 0$ if $p \nmid N$.

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3. Suppose N is squarefree and $p|N$. Then using Atkin-Lehner operators, all cusps can be moved to ∞. An easy calculation now shows that:

$$\mathrm{val}_p(f|_{\mathfrak{c}}) = \begin{cases} -\frac{k}{2} & \text{if } \mathrm{val}_p(L) = 0, \\ 0 & \text{if } \mathrm{val}_p(L) = 1. \end{cases}$$

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4. Nothing much previously known for general N. Some generic bounds exist due to Conrad using intersection theory on regular stacky surfaces, but are quite weak and have other issues.
5. For the general case, it suffices (thanks to AL operators) to restrict to cusps of denominator L such that $L^2|N$.

Examples

$N = 2^3 \cdot 3,\ k = 2,\ p = 2$

$f = q - q^2 + q^4 + q^5 + 2q^7 + \dots$

$f|_2\left(\begin{smallmatrix} 1 & 1 \\ 2 & 3 \end{smallmatrix}\right) = \frac{1}{6}\left(iq^{\frac{1}{6}} + iq^{\frac{1}{2}} - 2iq^{\frac{5}{6}} + \dots\right).$

So $\mathrm{val}_2(f|_{1/2}) = -1$.

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$N = 2 \cdot 3^5,\ k = 2,\ p = 3$

$f = q - q^2 + q^4 + 3q^5 - 4q^7 + \dots$

$$f|_2\left(\begin{smallmatrix} 1 & -1 \\ 3 & -2 \end{smallmatrix}\right) = \frac{1}{54}\left(\zeta_{162}^{25} q^{\frac{1}{54}} + \zeta_{162}^{50} q^{\frac{2}{54}} + \zeta_{162}^{19} q^{\frac{4}{54}} + \dots\right)$$

$$f|_2\left(\begin{smallmatrix} 1 & 1 \\ 9 & 10 \end{smallmatrix}\right) = \frac{1}{6}\left(\zeta_{54}^{7} q^{\frac{1}{6}} + \zeta_{54}^{14} q^{\frac{1}{3}} + \zeta_{54} q^{\frac{4}{6}} + \dots\right).$$

So $\mathrm{val}_3(f|_{1/3}) = -3$, $\mathrm{val}_3(f|_{1/9}) = -1$.

Examples (contd.)

$N = 5^2$, $k = 4$, $p = 5$

$f = q + 4q^2 - 2q^3 + 8q^4 + + \dots$

$f|_4\left(\begin{smallmatrix} 1 & 0 \\ 5 & 1 \end{smallmatrix}\right) = \frac{1}{5}\left(\left(-4\zeta_5^3 - 3\zeta_5 - 3\right) q + \left(-12\zeta_5^2 - 16\zeta_5 - 12\right) q^2 + \dots\right).$

$\mathrm{val}_5(f|_{1/5}) = -1/2.$

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$N = 7^2$, $k = 4$, $p = 7$

$f = q - 5q^2 + 17q^4 - 45q^8 + \dots$

$$f|_4\left(\begin{smallmatrix} 1 & \\ 7 & 1 \end{smallmatrix}\right) = \frac{1}{7}\Big(\left(-2\zeta_7^5 - 4\zeta_7^4 - 6\zeta_7^3 - 8\zeta_7^2 - 3\zeta_7 - 5\right) q$$
$$+ \left(-30\zeta_7^5 + 10\zeta_7^4 - 20\zeta_7^3 - 15\zeta_7^2 - 10\zeta_7 - 5\right) q^2 + \dots \Big)$$

$\mathrm{val}_7(f|_{1/7}) = -1/6.$

Examples (contd.)

$N = 2^8 \cdot 3,\ k = 2,\ p = 2$

$f = q + q^3 + 4q^7 + \dots.$

$$f|_2 \left(\begin{smallmatrix} 1 & 1 \\ 2 & 3 \end{smallmatrix}\right) = \frac{1}{192}\left(\zeta_{128} q^{\frac{1}{192}} + \zeta_{128}^3 q^{\frac{3}{192}} + \dots\right)$$

$$f|_2 \left(\begin{smallmatrix} 1 & -1 \\ 4 & -3 \end{smallmatrix}\right) = \frac{1}{48}\left(\zeta_{64}^{15} q^{\frac{1}{48}} + \dots\right)$$

$$f|_2 \left(\begin{smallmatrix} 3 & 1 \\ 8 & 3 \end{smallmatrix}\right) = \frac{1}{12}\left(\zeta_{32}^{5} q^{\frac{1}{12}} + \dots\right)$$

$$f|_2 \left(\begin{smallmatrix} 5 & -1 \\ 16 & -3 \end{smallmatrix}\right) = \frac{1}{3}\left(2\zeta_{16}^{7} q^{\frac{2}{3}} + \dots\right)$$

$\mathrm{val}_7(f|_{1/2}) = -6,\ \mathrm{val}_7(f|_{1/4}) = -4, \mathrm{val}_7(f|_{3/8}) = -2,\ \mathrm{val}_7(f|_{5/16}) = 1.$

The main theorem

Theorem 1

For a newform f of weight k for $\Gamma_0(N)$, a prime p, and a cusp $\mathfrak{c}$ of denominator L, the quantity $\mathrm{val}_p(f|_{\mathfrak{c}})$ depends only *on f and $\mathrm{val}_p(L)$. For $0 \le \mathrm{val}_p(L) \le \frac{\mathrm{val}_p(N)}{2}$, we have the bounds $\mathrm{val}_p(f|_{\mathfrak{c}}) \ge$*

$$-\frac{k}{2}\left(\mathrm{val}_p(N) - 2\mathrm{val}_p(L)\right) + \begin{cases} 0 & \text{if } \mathrm{val}_p(L) = 0, \\ 0 & \text{if } \mathrm{val}_p(L) = 1,\ \mathrm{val}_p(N) > 2, \\ -\frac{1}{2} & \text{if } \mathrm{val}_p(L) = \frac{1}{2}\mathrm{val}_p(N) = 1, \\ 1 - \frac{1}{2}\mathrm{val}_p(L) & \text{otherwise.} \end{cases}$$

For $p = 2$, we get even stronger bounds...

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Theorem 1 (contd...)

If $p = 2$ we have the additional stronger bounds.

$$\mathrm{val}_2(f|_{\mathfrak{c}}) \geq -\frac{k}{2}\left(\mathrm{val}_p(N) - 2\mathrm{val}_p(L)\right)$$

$$+\begin{cases} 0 & \textit{if } \ \mathrm{val}_2(L) = \frac{1}{2}\mathrm{val}_2(N) = 1, \\ \frac{k}{2} & \textit{if } \ \mathrm{val}_2(L) = \frac{1}{2}\mathrm{val}_2(N) \in \{2,3,4\}, \\ \frac{k}{2} + 1 - \frac{1}{4}\mathrm{val}_2(N) & \textit{if } \ \mathrm{val}_2(L) = \frac{1}{2}\mathrm{val}_2(N) > 4, \\ 0 & \textit{if } \ \mathrm{val}_2(L) = 3, \ \mathrm{val}_2(N) > 6. \end{cases}$$

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- We have checked experimentally that our bounds are sharp for newforms associated to elliptic curves and $p \leq 17$.

An application to the Manin constant

The modularity theorem (Wiles-Taylor, B-C-D-T)

Given an elliptic curve $E/\mathbb{Q}$ of conductor N,

- (*E is modular*) There exists a newform f of weight 2 for $\Gamma_0(N)$ and with integral Fourier coefficients such that $a_f(p) = p + 1 - |E(\mathbb{F}_p)|$.
- (*E has a modular parametrization*) There is a surjection $\phi\colon X_0(N)_{\mathbb{Q}} \twoheadrightarrow E$.

Note: ϕ is not unique, so it is common to normalize ϕ to be optimal, that is, $\deg(\phi)$ to be the least possible.

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Recall: $v_2(N) \leq 8$, $v_3(N) \leq 5$, $v_p(N) \leq 2$ for $p > 3$.

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Theorem 2

For $\Gamma_1(N) \subset \Gamma \subset \Gamma_0(N)$, every surjection $\phi\colon (X_\Gamma)_{\mathbb{Q}} \twoheadrightarrow E$ satisfies $c_\phi \mid 6 \cdot \deg(\phi)$, and if N is cube-free or $\Gamma = \Gamma_1(N)$, then even $c_\phi \mid \deg(\phi)$.

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This is interesting because $\deg(\phi)$ is a has little in common with N. No apparent connection between the conditions $p^2 | N$ and $p | \deg(\phi)$.

A *very* brief sketch of proof of Theorem 2:

1. Using Theorem 1, we show that

$$\omega_f \text{ lies in the } \mathbb{Z}\text{-lattice } H^0(X_0(N)_{\mathbb{Z}}, \Omega) \subset H^0(X_0(N)_{\mathbb{Q}}, \Omega^1), \qquad (1)$$

   where Ω denotes the relative dualizing sheaf. (Arithmetic geometric considerations reduce this to certain bounds on the p-adic valuations of the denominators of the Fourier coefficients of f at *all* the cusps of $X_0(N)_{\mathbb{C}}$. Theorem 1 gives much stronger bounds than needed.)

2. Using above, we show that ω_f lies in an even *a priori* smaller lattice $H^0(\mathcal{J}_0(N), \Omega^1)$ that seems otherwise inaccessible. Here $\mathcal{J}_0(N)$ is the Néron model of the Jacobian $J_0(N)$.

3. Now Theorem 2 follows from the fact that the composition $\pi \circ \pi^\vee \colon E \to J_0(N) \to E$ is multiplication by $\deg(\phi)$.

For the rest of this talk I will focus on the proof of Theorem 1.

Recall Theorem 1:

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with sharper bounds for $p = 2$.

Fourier expansions and Whittaker models

In order to prove Theorem 1, for a cusp $\mathfrak{c} = \gamma\infty$ and a prime p, we want to prove lower bounds on

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Fourier coefficients at general cusps are subtle: e.g., the coefficients $a_f(n;\mathfrak{c})$ are not multiplicative. One way to understand $a_f(n;\mathfrak{c})$ is via the Whittaker model.

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- ϕ_f generates a cuspidal automorphic representation $\pi = \otimes_v \pi_v$.
- The global Whittaker newform $W_f(g) = \int_{\mathbb{Q}\backslash\mathbb{A}} \phi_f(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) g)\psi(-x)dx$ packages together all Fourier coefficients at all cusps. In particular, $a_f(r; \mathfrak{c}) = W_f(g_{r,\mathfrak{c}})$ for some explicit $g_{r,\mathfrak{c}} \in \mathrm{GL}_2(\mathbb{A})$.

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- The global Whittaker newform $W_f(g) = \int_{\mathbb{Q}\backslash\mathbb{A}} \phi_f((\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix})g)\psi(-x)dx$ packages together all Fourier coefficients at all cusps. In particular, $a_f(r;\mathfrak{c}) = W_f(g_{r,\mathfrak{c}})$ for some explicit $g_{r,\mathfrak{c}} \in \mathrm{GL}_2(\mathbb{A})$.
- On the other hand, $W_f(g) = \prod_v W_{\pi_v}(g_v)$, where $W_{\pi_v} : \mathrm{GL}_2(\mathbb{Q}_v) \to \mathbb{C}$ is the local Whittaker newform that depends only on the local representation π_v.

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An explicit relation

For a newform f of weight k for $\Gamma_0(N)$, a prime p, and a matrix $\gamma = \left(\begin{smallmatrix} a & * \\ L & * \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, $\mathfrak{c} = \frac{a}{L}$, with $L^2|N$, up to a root of unity:

$$a_f(r;\mathfrak{c}) = a_f(r_0)\left(\frac{r}{r_0 w(\mathfrak{c})}\right)^{k/2} \prod_{q|N} W_{\pi_q}\left(\begin{pmatrix} 0 & \frac{r}{N} \\ 1 & \frac{u_q}{L} \end{pmatrix}\right).$$

where r_0 is the N-free part of r, and $u_q \in \mathbb{Z}_q^\times$.

- **Upshot:** Proving lower bounds for $\mathrm{val}_p(f|_\mathfrak{c})$ reduce to proving lower bounds for $\mathrm{val}_p\left(W_{\pi_q}\begin{pmatrix} 0 & q^t \\ 1 & \frac{u_q}{q^\ell} \end{pmatrix}\right)$ for primes p and q both dividing N, $t \in \mathbb{Z},\ 0 \leq \ell \leq \frac{c(\pi_q)}{2},\ u_q \in \mathbb{Z}_q^\times$.

- **Upshot:** Proving lower bounds for $\mathrm{val}_p(f|_{\mathfrak{c}})$ reduce to proving lower bounds for $\mathrm{val}_p\left(W_{\pi_q}\left(\begin{smallmatrix} 0 & q^t \\ 1 & \frac{u_q}{q^\ell}\end{smallmatrix}\right)\right)$ for primes p and q both dividing N, $t \in \mathbb{Z},\ 0 \leq \ell \leq \frac{c(\pi_q)}{2},\ u_q \in \mathbb{Z}_q^\times$.
- Since $|x|_p = p^{-\mathrm{val}_p(x)}$, this is a *p-adic analogue* of the local sup-norm question of bounding $|W_{\pi_q}|_\infty$ in highly ramified cases. (Templier 2014, S. 2016, Assing 2019)
- The values of W_{π_q} at diagonal matrices are well-known, the key point is to access the non-diagonal elements.
- **Remark**: Any matrix g in $\mathrm{GL}_2(\mathbb{Q}_q)$ has a double coset representative in $N(F)gK_0(n)$ of the form $\left(\begin{smallmatrix} 0 & q^t \\ 1 & \frac{u}{q^\ell}\end{smallmatrix}\right)$ for $0 \leq \ell \leq n$; local Atkin–Lehner operators halve the range of ℓ.

To prove lower bounds for $\mathrm{val}_p\left(W_{\pi_q}\left(\begin{smallmatrix} 0 & q^t \\ 1 & \frac{u}{q^\ell}\end{smallmatrix}\right)\right)$ we refine and extend a method developed for the *sup-norm problem* (S. 2016- 2019, Assing 2018-2019, Assing–Corbett 2019,...).

The local functional equation (Jacquet–Langlands, 1972)

For a non-archimedean local field F, an infinite-dimensional representation π of $\mathrm{GL}_2(F)$, an element W in the local Whittaker model of π, and a character μ of $F^\times$, putting

$$Z(W, s, \mu) = \int_{F^\times} W(\left(\begin{smallmatrix} y & \\ & 1 \end{smallmatrix}\right))\mu(y)|y|^{s-\frac{1}{2}}\, d^\times y$$

$$\frac{Z(W, s, \mu)}{L(s, \pi \otimes \mu)}\varepsilon(s, \pi \otimes \mu) = \frac{Z(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \cdot W, 1-s, \mu^{-1})}{L(1-s, \pi \otimes \mu^{-1})}, \qquad (2)$$

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Above $\varepsilon(s,\pi)$ is the local GL_2 ϵ-factor (Jacquet–Langlands).

Using this, one can formulate a "basic identity" (S, 2016) that writes down $W_{\pi_q}(g_q)$ as an explicit linear combination of terms involving GL_2 and GL_1 ϵ-factors.

For example, if π is supercuspidal, the basic identity becomes

The basic identity for supercuspidal reps π

For a supercuspidal rep π of $\mathrm{PGL}_2(\mathbb{Q}_q)$, $u \in \mathbb{Z}_q^\times$, and $1 \leq \ell \leq \frac{c(\pi)}{2}$.

$$W_\pi \begin{pmatrix} 0 & q^t \\ 1 & \frac{u}{q^\ell} \end{pmatrix} = (1 - q^{-1})^{-1} \; q^{-\frac{\ell}{2}} \sum_{\substack{c(\mu)=\ell \\ c(\mu\pi)=-t}} \varepsilon(1/2, \mu) \; \varepsilon(1/2, \mu^{-1}\pi)\mu(u). \quad (3)$$

For other representations, the basic identity takes a similar (though slightly more complicated) shape. The resulting formulae were written by me in some cases (S, 2016 - 2018) and in all cases by Assing in his thesis (2019).

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So we need to solve the problem of computing p-adic valuations of ε-factors of representations of $\mathrm{GL}_r(\mathbb{Q}_q)$ where $r = 1, 2$.

The case $q \neq p$

Theorem 3

For a finite extension $F/\mathbb{Q}_q$, an infinite-dimensional ramified representation π of $\mathrm{GL}_2(F)$ associated to a holomorphic newform, and a matrix $g \in \mathrm{GL}_2(F)$, we have $W_\pi(g) \in \overline{\mathbb{Z}}\left[\frac{1}{q}\right]$.
In particular, if $p \neq q$, then $\mathrm{val}_p(W_\pi(g)) \geq 0$.

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This relies on a formula for the Whittaker newvector in terms of a family of nonarchimedean ${}_2F_1$ hypergeometric integrals (Assing 2019; also unpublished works of Templier (2012) and Hu (2016)).

Sketch of proof of Theorem 3 (assuming above-mentioned formula)

Suppose G compact group, $K \subseteq G$ of finite index, $\mathrm{vol}(K) \in R$. Let $f : G \mapsto R$ be a right-K-invariant function. Then $\int_G f(g)dg \in R$.

So we are reduced to the case $q = p$.

The case $q = p$

So the next problem is: Let F be a finite extension of $\mathbb{Q}_p$. Understand the p-adic valuations of $\varepsilon(1/2, \mu)$ and $\varepsilon(1/2, \mu \otimes \pi)$ where μ is a finite order character of $F^\times$ and π be an infinite-dimensional, irreducible, unitary representation of $\mathrm{PGL}_2(F)$.

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- If π is principal series, we need to also assume that it comes from a global holomorphic newform (otherwise we cannot expect good results).
- Note: $\varepsilon(1/2, \mu)$ and $\varepsilon(1/2, \mu \otimes \pi)$ are **algebraic numbers of absolute value 1**, but are **not** necessarily roots of unity.

The case of GL_1

- The GL_1 ϵ-factors defined by Tate are closely related to classical Gauss sums.
- For a classical Gauss sum, there is a well-known result (Stickelberger's congruence) that gives its p-adic valuation.

Theorem 4

For a finite extension $F/\mathbb{Q}_p$, and a character $\chi\colon F^\times \to \mathbb{C}^\times$ of finite order,

1. *if $a(\chi) = 1$, then,*

$$\mathrm{val}_p(\varepsilon(\frac{1}{2},\chi)) = -\frac{[\mathbb{F}_F : \mathbb{F}_p]}{2} + \frac{s(\chi)}{p-1},\ 0 \leq s(\chi) \leq (p-1)[\mathbb{F}_F/\mathbb{F}_p];$$

2. *if $\chi^2 = 1$ or $a(\chi) > 1$, then $\varepsilon(\frac{1}{2},\chi)$ is a root of unity, and so*

$$\mathrm{val}_p(\varepsilon(\frac{1}{2},\chi)) = 0.$$

A classification of infinite-dimensional, irreducible, unitary representation of $\mathrm{GL}_2(F)$ and trivial central character.

1. Principal series representations
2. Special representations (twists of Steinberg)
3. Supercuspidal representations:
    a. Dihedral supercuspidal
    b. Non-dihedral supercuspidal (can only occur if $p = 2$)

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In cases 1, 2 and 3a, one can write the GL_2 ε-factor in terms of GL_1 $\varepsilon-$factors. So the problem here reduces to one we have solved.

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Analysis of non-dihedral representations

There are exactly 16 representations of Type 3b. Using the Local Langlands correspondence and the basic identity we write down $W_\pi(g)$ exactly in each case, from which the required bounds follow.

- We now know how to estimate the p-adic valuations of GL_r-ϵ-factors for $r = 1, 2$.
- The basic identity expresses $W_{\pi_p}\begin{pmatrix} 0 & p^t \\ 1 & \frac{u_p}{p^\ell} \end{pmatrix}$ as an explicit finite sum involving the above.
- This allows us to prove our main local result, which gives sharp p-adic bounds for $\mathrm{val}_p(W_{\pi_p}(..)$ in all cases.

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Here is a special case of our main local result:

A special case of our local theorem

Theorem 5

Let p be odd and $F/\mathbb{Q}_p$ a finite extension. Let π be a supercuspidal representation of $\mathrm{GL}_2(F)$, with trivial central character and $c(\pi) = n > 2$. For $0 \leq \ell \leq n/2$ and $u \in \mathcal{O}_F$,

$$\mathrm{val}_p(W_\pi(\left(\begin{smallmatrix} 0 & p^t \\ 1 & up^{-\ell} \end{smallmatrix}\right))) \geq \begin{cases} 0 & \textit{if } \ell = 0, 1, \\ [\mathbb{F}_F : \mathbb{F}_p]\left(1 - \frac{\ell}{2}\right) & \textit{otherwise.} \end{cases}$$

- Our main local theorem gives such bounds (with *lots* of subcases) covering all representations and conductors.
- If $p = 2$, we only do the case $F = \mathbb{Q}_2$.
- We get stronger bounds for $\mathbb{Q}_2$ by exploiting additional parity cancellation in sums of ϵ-factors.

Now, Theorem 1 follows as described earlier...

That is, we combine the local bounds on $\mathrm{val}_p(W_\pi(\left(\begin{smallmatrix} 0 & p^t \\ 1 & up^{-\ell} \end{smallmatrix}\right)))$ given by (the general version of) Theorem 5 with

$$a_f(r;\mathfrak{c}) = a_f(r_0)\left(\frac{r}{r_0 w(\mathfrak{c})}\right)^{k/2}\prod_{q|N} W_{\pi_q}\left(\left(\begin{smallmatrix} 0 & \frac{r}{N} \\ 1 & \frac{uq}{L} \end{smallmatrix}\right)\right)$$

to obtain the sharp lower bounds for $\mathrm{val}_p(f|_\mathfrak{c})$ for holomorphic newforms f at each cusp $\mathfrak{c}$, which is the content of Theorem 1.

Theorem 1

For a newform f of weight k for $\Gamma_0(N)$, a prime p, and a cusp $\mathfrak{c}$ of denominator L, the quantity $\mathrm{val}_p(f|_{\mathfrak{c}})$ depends only on f and $\mathrm{val}_p(L)$. For $0 \leq \mathrm{val}_p(L) \leq \frac{\mathrm{val}_p(N)}{2}$, we have the bounds $\mathrm{val}_p(f|_{\mathfrak{c}}) \geq$

$$-\frac{k}{2}\left(\mathrm{val}_p(N) - 2\mathrm{val}_p(L)\right) + \begin{cases} 0 & \text{if } \mathrm{val}_p(L) = 0, \\ 0 & \text{if } \mathrm{val}_p(L) = 1,\ \mathrm{val}_p(N) > 2, \\ -\frac{1}{2} & \text{if } \mathrm{val}_p(L) = \frac{1}{2}\mathrm{val}_p(N) = 1, \\ 1 - \frac{1}{2}\mathrm{val}_p(L) & \text{otherwise.} \end{cases}$$

For $p = 2$, we get even stronger bounds...

The main theorem

Theorem 1 (contd...)

If $p = 2$ we have the additional stronger bounds.

$$\mathrm{val}_2(f|_{\mathfrak{c}}) \geq -\frac{k}{2}\left(\mathrm{val}_p(N) - 2\mathrm{val}_p(L)\right)$$

$$+\begin{cases} 0 & \textit{if } \ \mathrm{val}_2(L) = \frac{1}{2}\mathrm{val}_2(N) = 1, \\ \frac{k}{2} & \textit{if } \ \mathrm{val}_2(L) = \frac{1}{2}\mathrm{val}_2(N) \in \{2, 3, 4\}, \\ \frac{k}{2} + 1 - \frac{1}{4}\mathrm{val}_2(N) & \textit{if } \ \mathrm{val}_2(L) = \frac{1}{2}\mathrm{val}_2(N) > 4, \\ 0 & \textit{if } \ \mathrm{val}_2(L) = 3, \ \mathrm{val}_2(N) > 6. \end{cases}$$

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Thank you!