HILBERT MODULAR FORMS OF WEIGHT 1/2 AND THETA FUNCTIONS

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Abstract. Serre and Stark found a basis for the space of modular forms of weight 1/2 in terms of theta series. In this paper, we generalize their result - under certain mild restrictions on the level and character - to the case of weight 1/2 Hilbert modular forms over a totally real field of narrow class number 1. The methods broadly follow those of Serre-Stark; however we are forced to overcome technical difficulties which arise when we move out of $\mathbb{Q}$.

Introduction

0.1. **The problem and the main result.** Shimura, at the end of his fundamental paper [7] on elliptic modular forms of half-integral weight, mentioned certain questions that were open at the time: one of them asked whether every modular form of weight 1/2 is a linear combination of theta series in one variable. This was answered in the affirmative by Serre–Stark [6] who gave an explicit basis for the space of modular forms of weight 1/2, level N and character ψ in terms of certain theta series. These theta series are denoted by $\theta_{\chi,t}$ where χ is a primitive Dirichlet character and t a positive integer so that χ and t are related in a precise manner to N and ψ. Such an explicit result has several nice applications, see for instance Tunnell's work [11] on the ancient congruent number problem.

It seems natural to generalize the Serre-Stark theorem to fields other than $\mathbb{Q}$, that is, to find an explicit basis in terms of theta series for *Hilbert modular forms* of weight $(\frac{1}{2}, \frac{1}{2}, ...\frac{1}{2})$. In this paper we achieve that in the case of a totally real field F of narrow class number 1 when the level $\mathbf{c}$ and character ψ of the form have certain nice properties. In particular we assume that no prime dividing $\mathbf{c}$ splits in the extension $F/\mathbb{Q}$ and that the Dirichlet character ψ of the form is trivial at the units (or equivalently, the corresponding finite order Hecke character is trivial at all infinite places). Under these assumptions we prove that the space of Hilbert modular forms of weight $(\frac{1}{2}, \frac{1}{2}, ...\frac{1}{2})$, level $\mathbf{c}$ and character ψ has a basis consisting of theta series that are almost identical to the ones in Serre-Stark's theorem.

We note here that Shimura proved (see Theorem 1.1) that the space of weight 1/2 Hilbert modular forms of *all levels* is *spanned* by certain theta series; however his results do not seem to give a *basis*, nor do they appear to apply to a *particular level*. Also, as noted by Deligne in a letter (appended at the end of [6]) the problem can be attacked using the tools of representation theory. This was carried through successfully by Gelbart–Piatetski-Shapiro [5]; however their result, like Shimura's, only finds a spanning set and also does not consider the levels.

Date: March 2008.

The authors wish to thank their advisor D. Ramakrishnan for guidance and helpful discussions.

We now briefly state the main result. Let F be a totally real number field of narrow class number 1 and degree n over $\mathbb{Q}$. Let R be its ring of integers, $R^+ \subset R$ the subset of totally positive elements and U the subgroup of units in R. For an ideal $\mathbf{c}$ of R that is divisible by 4 and all of whose prime divisors are non-split, and a primitive Dirichlet character ψ trivial on U, we let $M(\mathbf{c}, \psi)$ denote the space of Hilbert modular forms over F of parallel weight 1/2, level $\mathbf{c}$ and character ψ. For a primitive Dirichlet character χ trivial on U and of conductor $r(\chi)$, and an element $t \in R^+$, we define the theta-series $\theta_{\chi,t}$ on the n-fold product of the upper half plane by

$$\theta_{\chi,t}(z) = \sum_{x \in R} \chi^{-1}(x) e^{\pi i tr(x^2 z)}.$$

Then our main theorem says the following.

A basis for $M(\mathbf{c}, \psi)$ is obtained by taking all the theta-series $\theta_{\chi,t}$ where we let t vary over a set of representatives of R^+/U^2 and let χ satisfy, in addition to the conditions mentioned above, the following:

(1) $4r(\chi)^2 t$ *divides* $\mathbf{c}$*,*
(2) $\psi = \chi\epsilon_t$ *where* ϵ_t *is the character associated to the quadratic extension* $F(\sqrt{t})$*.*

A few words about our methods. Though our techniques are similar to those of [6], there are certain complications which arise because we are no longer dealing with $\mathbb{Q}$; as a result many of the proofs of [6] do not extend to our case easily. Here are two main points of difference:

First, in section 4 we prove various properties of certain operators (such as the symmetry operator) that are crucial to the theory of newforms for Hilbert modular forms of parallel weight 1/2. In [6] these properties can be easily checked by hand and are left as exercises; however that is not the case here because we *do not* have a simple closed formula for the automorphy factor. So we use an expression for the automorphy factor from Garrett's book [4] and certain relations due to Shimura (and do some messy computations) to prove these properties. Furthermore we have to be very careful in the way we normalize these operators (and take into consideration the fact that the different of the field is no longer equal to 1) so that things work out.

Secondly, the proof of the crucial Theorem 6.2 does not quite go through in a manner similar to [6]; the clever divisibility argument at the end of that proof breaks down here because of primes above 2. We use a completely different method to get around this conundrum; we essentially use the fact that the size of the Fourier coefficients is bounded by Shimura's work.

Thus, the basis problem for the Hilbert modular case is not a completely straightforward extension of [6], which, we hope, justifies this article. Besides, we indicate, in a short section at the end, a motivation for solving this problem, by pointing out two potential applications which we hope to take up elsewhere.

0.2. **Structure of the paper.** In Section 1 we lay down notation, give some important definitions and results that will be used throughout the paper, state an important result due to Shimura and give the precise statement of our main theorem.

In Section 2 we define the Hecke operators and write down their action on Fourier coefficients.

Section 3, titled 'Easy pickings' is the analogue of [6, Section 5]. All the proofs carry over *mutatis mutandis* from there. We have included them for completeness.

Section 4 is similarly analogous to [6, Subsection 3.4]. In this section we define some important operators (there are some differences from the corresponding definitions in

[6] which arise because our definition of a modular form is not *quite* the same as Serre-Stark's) and prove the same results as in there. However the calculations now are of a higher order of difficulty than in [6] because, unlike in the classical case, there is no simple formula for the automorphy factor. As a result the proofs are more technical. This is probably the hardest part of the paper, involving messy computations.

In Section 5 we outline the theory of newforms for our purposes. The proofs are but formal consequences of the results of the previous two sections and essentially identical to the corresponding proofs in [6]. Therefore we do not include them.

In Section 6, we define the L-series and use it to characterize a newform. Using that, we prove our main theorem. At the end of this section, we illustrate our theorem by writing down bases for the spaces of weight 1/2 Hilbert modular forms over $\mathbb{Q}(\sqrt{2})$ for various levels.

Finally, in Section 7 we mention some potential applications of our work.

1. Preliminaries

1.1. **Notation.** Let F be a totally real number field, R its ring of integers, D its discriminant and δ its different. By abuse of notation we also use δ to denote a fixed totally positive generator of the different. We assume that F has narrow class number one and we let n denote the degree of $F/\mathbb{Q}$. Let the group of units of F be denoted by U and the group of totally positive units by U^2 (since the field is of narrow class number one, all totally positive units are squares). For any $t \in F$, we use the notation $t >> 0$ to mean that t is totally positive.

We denote the adelization of F by $F_{\mathbf{A}}$ and the ideles by $F_{\mathbf{A}}^{\times}$. For any $x \in F$ let $N(x)$ denote its norm over $\mathbb{Q}$. For an ideal $\mathbf{m} \subset R$, we will let $N(\mathbf{m})$ denote the cardinality of $R/\mathbf{m}$. Let ∞ denote the set of Archimedean places of F and $\mathbf{f}$ denote the finite places. For $g \in F_{\mathbf{A}}^{\times}$ we denote $g_{\mathbf{m}} = \prod_{v|\mathbf{m}} g_v$ and $g_\infty = \prod_{v\in\infty} g_v$. For $v \in \infty$ we denote the positive elements of F_v by F_v^+. By F_∞° we mean the connected component at infinity of the identity, i.e.

$$F_\infty^\circ = \prod_{v\in\infty} F_v^+ \simeq (\mathbb{R}^+)^n.$$

Let $\mathbb{H}^n$ (resp. $\mathbb{C}^n$) denote the n-fold product of the upper half plane (resp. complex plane). For $z = (z_1, .., z_n)$ in $\mathbb{C}^n$ or $\mathbb{R}^n$ and any $\alpha \in \mathbb{R}$ we put

$$z^\alpha = \prod_{i=1}^{n} z_i^\alpha, \quad e(z) = \prod_{i=1}^{n} e^{2\pi i z_i}, \quad N(z) = \prod_{i=1}^{n} z_i, \quad tr(z) = \sum_{i=1}^{n} z_i.$$

We also use the symbol $e(z)$ for $z \in F$ using the n embeddings of F in $\mathbb{R}$.

Furthermore, for any prime(i.e. a finite place) p in $\mathbb{R}$ we define the character e_p on F as follows: For $x \in F$ let

$$e_p(x) = e^{-2\pi i y}$$

where $y \in \cap_{q\neq p'}(\mathbb{Z}_q \cap \mathbb{Q})$, $y - Tr_{F_p/\mathbb{Q}_{p'}}(x) \in \mathbb{Z}_{p'}$. Here q is any prime in $\mathbb{Z}$ and p' is the prime below p, that is, $p' = p \cap \mathbb{Q}$.

1.2. **Conventions on characters.** Let $\mathbf{m}$ be an ideal of R. A Dirichlet character mod $\mathbf{m}$ is a function ϕ from R to the unit circle such that:

1. There exists a homomorphism $\overline{\phi}$ from the finite group $(R/\mathbf{m})^\times$ to the unit circle such that for any a in R that is relatively prime to $\mathbf{m}$ we have $\phi(a) = \overline{\phi}(\overline{a})$
2. For any a that shares a common factor with $\mathbf{m}$, we have $\phi(a) = 0$.

Such a ϕ is called *primitive* if $\overline{\phi}$ does not factor through $(R/\mathbf{m}')^\times$ for some proper divisor $\mathbf{m}'$ of $\mathbf{m}$.

By a Hecke character of F we mean a character of $F_{\mathbf{A}}^\times$ which is trivial on $F^\times$ and has values in the unit circle. For a Hecke character ψ and any place v, ψ_v denotes the restriction of ψ to $F_v^\times$. For an ideal $\mathbf{c}$ of R, let

$$\psi_{\mathbf{c}} = \prod_{v|\mathbf{c}} \psi_v,$$

$$\psi_{\mathbf{f}} = \prod_{v|f} \psi_v$$

and

$$\psi_\infty = \prod_{v\in\infty} \psi_v.$$

The conductor of ψ always refers to its finite part and is denoted by $r(\psi)$. In a mild abuse of notation we will use, for g in $F^\times$ or even $F_{\mathbf{A}}^\times$, $\psi_{\mathbf{c}}(g)$ (resp. $\psi_\infty(g)$) to mean $\psi_{\mathbf{c}}(g_{\mathbf{c}})$ (resp. $\psi_\infty(g_\infty)$). Also let ψ^* denote the corresponding character on the ideals as defined in [10, p. 238]. In particular, if I is an ideal of R generated by s and $(I, r(\psi)) = 1$, we have

$$\psi^*(I) = \overline{\psi_{\mathbf{c}}(s)\psi_\infty(s)}$$

for any $\mathbf{c}$ divisible by $r(\psi)$ with $(I, \mathbf{c}) = 1$. We will use the notation $\psi^*(a)$ for $a \in R$ to denote $\psi^*((a))$.

For any $\tau \in F$ let ϵ_τ denote the Hecke character of F corresponding to $F(\tau^{1/2})/F$.

Comment: It is well known that any *finite order* Hecke character ψ of F gives rise to a primitive Dirichlet character mod $r(\psi)$. This correspondence is bijective. Moreover, for such a finite order Hecke character ψ, we have $\psi_\infty(g) = \prod_{v\in\infty} \mathrm{sgn}(g_v)^{e_v}$ with each $e_v = 0$ or 1. Thus ψ_∞ is trivial if and only if each $e_v = 0$. It can be checked that this happens if and only if the corresponding Dirichlet character is trivial on the units of R.

1.3. **Conventions on modular forms.** *In the rest of this paper, unless mentioned otherwise, we will use the term Hecke character to mean finite order Hecke character.*

Given a 2×2 matrix $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we write $a = a_\alpha,\ b = b_\alpha,\ c = c_\alpha$ and $d = d_\alpha$.

Let $G = SL_2(F)$. We will consider weight $(\frac{1}{2}, ..., \frac{1}{2})$ modular forms on the congruence subgroups of G.

For any two fractional ideals $\mathfrak{f}, \mathfrak{g}$ of R, let $\Gamma[\mathfrak{f}, \mathfrak{g}]$ denote the subgroup of G consisting of matrices γ such that $a_\gamma,\ d_\gamma \in R,\ b_\gamma \in \mathfrak{f},\ c_\gamma \in \mathfrak{g}$. Let $\mathbf{D}$ denote the group $\Gamma[2\delta^{-1}, 2\delta]$. A congruence subgroup is a subgroup of $\mathbf{D}$ that contains a principal congruence subgroup $\Gamma(N)$ for some integer N, where

$$\Gamma(N) = \{\gamma \in SL_2(R) : \gamma \equiv I \pmod{N}\}.$$

For any two integral ideals $\mathbf{c}, \mathbf{d}$, with $4|\mathbf{c}$ we use the notation

$$\Gamma_{\mathbf{c},\mathbf{d}} = \Gamma[2\delta^{-1}\mathbf{d}, 2^{-1}\delta\mathbf{c}]$$

and

$$\Gamma_{\mathbf{c}} = \Gamma[2\delta^{-1}, 2^{-1}\delta\mathbf{c}].$$

Note that $\Gamma_{\mathbf{c},\mathbf{d}}$ and $\Gamma_{\mathbf{c}}$ are congruence subgroups.

For $\gamma \in \mathbf{D}$, $z \in \mathbb{H}^n$, let $h(\gamma, z)$ denote the automorphy factor $\frac{\theta(\gamma z)}{\theta(z)}$, where

$$\theta(z) = \sum_{x \in R} e(x^2 z/2).$$

A generalization of this automorphy factor is introduced in [8] where many of its properties are proved.

Now, let $\gamma \in \mathbf{D}$, and f be a holomorphic function on $\mathbb{H}^n$. We use the notation

$$(f \parallel \gamma)(z) = h(\gamma, z)^{-1} f(\gamma(z)).$$

Suppose now that $\mathbf{c}$ is an ideal as above and ψ is a Hecke character whose conductor divides $\mathbf{c}$ and $\psi_\infty(-1) = 1$. Let $M(\mathbf{c}, \psi)$ denote the space of modular forms of weight $(\frac{1}{2}, ... \frac{1}{2})$ on $\Gamma_{\mathbf{c}}$ with character ψ. In other words, $M(\mathbf{c}, \psi)$ is the set of holomorphic functions f on $\mathbb{H}^n$ satisfying

$$f \parallel \gamma = \psi_{\mathbf{c}}(a_\gamma) f$$

for all $\gamma \in \Gamma_{\mathbf{c}}$. Note that our definition follows [9] (and is slightly different from [6] where f satisfies $f \parallel \gamma = \psi_{\mathbf{c}}(d_\gamma) f$).

For each such $\mathbf{c}$ let $M^1(\mathbf{c})$ be the union of all $M(\mathbf{c}, \psi)$ with ψ varying over all Hecke characters with conductor dividing $\mathbf{c}$ and $\psi_\infty(-1) = 1$. Let M^1 be the union of all the $M^1(\mathbf{c})$ as $\mathbf{c}$ varies over the integral ideals of R divisible by 4. Finally, let M be the space of all weight $(\frac{1}{2}, ... \frac{1}{2})$ modular forms on congruence subgroups of G. Clearly for any such $\mathbf{c}, \psi$, $M(\mathbf{c}, \psi) \subset M^1(\mathbf{c}) \subset M^1 \subset M$.

Any $f \in M$ has a *Fourier expansion*

$$f(z) = \sum_{\xi \in F} a(\xi) e(\xi z/2).$$

We call $a(\xi)$ the Fourier coefficient for the place ξ.

If f belongs to M^1 then by [9, p. 780], the Fourier coefficients associated to places outside R are zero. Thus f has a Fourier expansion

$$f(z) = \sum_{\xi \in R} a(\xi) e(\xi z/2).$$

We are interested in the question of finding a basis for each of the spaces $M(\mathbf{c}, \psi)$.

1.4. **Theta functions.** Let η be a locally constant function on F, i.e. a complex valued function for which there exists two $\mathbb{Z}$-lattices L and M in F such that $\eta(x) = 0$ for x not in L and $\eta(x)$ depends only on x modulo M.

The following alternate criterion will be useful.

Proposition 1.1. *A function $\eta : F \to \mathbb{C}$ is locally constant if and only if there exist integers m, n such that $\eta(x) = 0$ for x not in $\frac{1}{m}R$ and $\eta(x)$ depends only on x mod (n).*

Proof. Any $\mathbb{Z}$-lattice contains (n) and is contained in $(\frac{1}{m})$ for some m, n. □

Let $\mathfrak{L}(F)$ denote this space of locally constant functions. We define the function θ_η on $\mathbb{H}^n$ by

$$\theta_\eta(z) = \sum_{\xi \in F} \eta(\xi) e(\xi^2 z/2).$$

We have the following proposition.

Proposition 1.2. *Let $\eta \in \mathfrak{L}(F)$. Then $\theta_\eta \in M$.*

Proof. This follows from [9, Lemma 4.1]. Indeed the proof there makes it clear that θ_η is a modular form for the largest congruence group contained in $\{\alpha \in D, ^\alpha \eta = \eta\}$, where $^\alpha\eta$ denotes the action of α on η as described in [9, p. 775]. □

1.5. **An important example.** The following example from [9] introduces the theta series that is fundamental to this paper.

Example 1 ([9] , p. 784-785)**.** *Let χ be a Hecke character of F of conductor f such that $\chi_\infty(-1) = 1$. Suppose ω_v denotes the characteristic function of R_v and let*

$$\eta(x) = \prod_{v \in \mathbf{f}} \eta_v(x_v)$$

where:

- $\eta_v = \omega_v$ *if* $v \nmid f$
- $\eta_v = \chi_v(t)^{-1}$ *if* $v \mid f$ *and* $\mid t \mid_v = 1$
- $\eta_v = 0$ *if* $v \mid f$ *and* $\mid t \mid_v \neq 1$

Then $\theta_\eta(z) \in M(4f^2, \chi)$.

For any Hecke character χ of F such that $\chi_\infty(-1) = 1$ we define θ_χ to equal θ_η where η is as in the above example. Thus $\theta_\chi \in M(4r(\chi)^2, \chi)$.

For any totally positive $t \in F$ let $\theta_{\chi,t}(z) := \theta_\chi(tz)$. We have $\theta_{\chi,t} \in M(\mathbf{c}, \psi)$ whenever $(4r(\psi)^2 t) \mid \mathbf{c}$ and $\psi = \chi\epsilon_t$. Refer to Lemma 4.1 for a proof of this fact. Similarly, for any function $\eta \in \mathfrak{L}(F)$ let $\theta_{\eta,t}(z) := \theta_\eta(tz)$.

1.6. **Two generating sets.** The following important theorem is due to Shimura and is contained in [9].

Theorem 1.1 (Shimura)**.** *M is spanned by the functions $\theta_{\eta,t}$ for $t \in F$ totally positive and $\eta \in \mathfrak{L}(F)$.*

What about the space of forms M^1?

We make the following preliminary observations:

Any $f \in M$, by the above theorem, can be written as

$$f(z) = \theta_{\eta_1}(t_1 z) + \theta_{\eta_2}(t_2 z) + .. + \theta_{\eta_k}(t_k z) \tag{1.6.1}$$

with $0 << t_i \in F$.

Replacing each $\eta_i(z)$ by $\frac{\eta_i(z)+\eta_i(-z)}{2}$, we may assume that $\eta_i(z) = \eta_i(-z)$. Note that this does not change the functions θ_{η_i}.

Also, we may assume that the t_i are distinct mod $(F^*)^2$. For, if $t_1 = s^2 t_2$, say, then

$$\theta_{\eta_1}(t_1 z) + \theta_{\eta_2}(t_2 z) = \theta_\eta(t_2 z)$$

where $\eta(z) = \eta_1(z/s) + \eta_2(z)$, and so we may combine those two summands into a single one.

Furthermore, if $\eta_i(\xi) = 0$ for ξ not in $(\frac{1}{m})R$, then $\eta_i(\frac{\xi}{m}) = 0$ for ξ not in R. Moreover, observe that $\theta_{\eta_i}(t_i z) = \theta_{\eta_i'}(t_i z/m^2)$ where $\eta_i'(z) = \eta_i(z/m)$. So, in (1.6.1) we may assume that each η_i is 0 outside R. We can now give a set of generators for M^1.

Theorem 1.2. *M^1 is spanned by the functions $\theta_{\eta,t}$ where $t \in R$ is totally positive, and $\eta \in \mathfrak{L}(F)$ satisfies $\eta(z) = 0$ if z does not belong to R.*

Proof. Any $f \in M^1$, by the above comments can be written as

$$f(z) = \theta_{\eta_1}(t_1 z) + \theta_{\eta_2}(t_2 z) + .. + \theta_{\eta_k}(t_k z) \tag{1.6.2}$$

where $0 << t_i \in F$ are distinct mod $(F^*)^2$ and $\eta_i \in \mathfrak{L}(F)$ are 0 outside R.

Then, because the t_i are distinct mod $(F^*)^2$ the various $\theta_{\eta_i}(t_i z)$ contribute distinct terms to the Fourier expansion of f. However only the Fourier coefficients corresponding to elements of R can be non-zero.

So for each i we must have $\eta_i(\xi) = 0$ whenever $\xi^2 t_i$ not in R. For a fixed t_i, the set of $\xi \in R$ such that $\xi^2 t_i \in R$ is an ideal, hence generated by some h. Put $\eta_i^{'}(z) = \eta_i(hz)$. Then $\theta_{\eta_i}(t_i z) = \theta_{\eta_i^{'}}(t_i h^2 z)$. Thus replacing η_i by $\eta_i^{'}$ and t_i by $t_i h^2$ we see that $\eta_i^{'}$ is still 0 outside R, but now $t_i h^2$ also belongs to R.

In other words we have shown that in 1.6.2, under the assumption $f \in M^1$ we can have $0 << t \in R$, and η is 0 outside R.

Conversely any such sum is in M^1 by [9, Proposition 3.2] and [4, p. 154].

This completes the proof. □

Corollary 1.1. *Let $f(z) = \sum_{\xi \in R} a(\xi) e(\xi z/2)$ be an element of $M(\mathbf{c}, \psi)$ for some $(\mathbf{c}, \psi)$. Then there is a constant C_f such that $\mid a(\xi) \mid < C_f$ for all $\xi \in R$.*

Proof. By the above theorem, it suffices to prove that $\theta_{\eta,t}$ has this property. But that follows easily from Proposition 1.1. □

1.7. **Statement of the main theorem.** Let R^+ denotes the set of totally positive elements in R. Fix a complete set of representatives T of R^+/U^2.

Suppose $\mathbf{c}$ is an integral ideal and ψ a Hecke character. Define $\Omega(\mathbf{c}, \psi)$ to be the set of pairs (χ, t) such that:

(1) χ is a Hecke character with χ_∞ trivial and $t \in T$.
(2) $4r(\chi)^2 t$ divides $\mathbf{c}$.
(3) $\psi = \chi \epsilon_t$.

Recall the definition of $\theta_{\chi,t}$ from Section 1.5. Our main Theorem is as follows:

Theorem 1.3. *Suppose $\mathbf{c}$ is an integral ideal divisible by 4. Let ψ be a Hecke character of F such that ψ_∞ is trivial and $r(\psi)$ divides $\mathbf{c}$. Assume that any prime ideal $\mathfrak{p}$ dividing $\mathbf{c}$ has the property that $\mathfrak{p}$ is the unique prime ideal of R that lies above $\mathfrak{p} \cap \mathbb{Z}$. Then the functions $\theta_{\chi,t}$ with $(\chi, t) \in \Omega(\mathbf{c}, \psi)$ form a basis of $M(\mathbf{c}, \psi)$.*

We prove this theorem in section 6.

2. Hecke operators

2.1. **Some definitions.** Let $GL_2^+(F)$ denote the subgroup of $GL_2(F)$ consisting of matrices whose determinant is totally positive. Let $\mathcal{G}$ denote the group extension of $GL_2^+(F)$ consisting of pairs $[A, \phi(z)]$ where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(F)$ and $\phi(z)$ is a holomorphic function on $\mathbb{H}^n$ satisfying $\phi(z)^2 = tN(\det A)^{-1/2} \prod (c^{(i)} z_i + d^{(i)})$ where $A^{(i)} = \begin{pmatrix} a^{(i)} & b^{(i)} \\ c^{(i)} & d^{(i)} \end{pmatrix}$ are the various embeddings of A in $GL_2(\mathbb{R})$ and t is a complex number with $\mid t \mid = 1$. The group law in $\mathcal{G}$ is given by $[A, \phi(z)][B, \psi(z)] = [AB, \phi(Bz)\psi(z)]$.

The group $\mathcal{G}$ acts on the *right* of the space of holomorphic functions on $\mathbb{H}^n$ as follows: For a holomorphic function f on $\mathbb{H}^n$ define $f \mid [A, \phi(z)] = \phi(z)^{-1} f(Az)$. Note also that the group D embeds in $\mathcal{G}$ via $A \to [A, h(A,z)$. Furthermore, we have $(f \parallel A)(z) = f \mid [A, h(A,z)]$.

For $\gamma = w_1 t w_2$ where $w_1, w_2 \in \mathbf{D}$ and $t = \begin{pmatrix} 1/a & 0 \\ 0 & a \end{pmatrix}$ for some $a \in R$, define $J_\Xi(\gamma, z) = h(w_1 w_2, z)$. The quantities $J_\Xi(\gamma, z)$ and $h(\gamma, z)$ coincide whenever $\gamma \in \mathbf{D}$. We also recall from [9] that:

(1) $J_\Xi\left(\begin{pmatrix} 1/p & 2b/(\delta p) \\ 0 & p \end{pmatrix}, z\right) = N(p)^{1/2}$ for a prime p and element b in R.

(2) $J_\Xi\left(\begin{pmatrix} 1 & 2h/(\delta p) \\ 0 & 1 \end{pmatrix}, z\right) = N(p)^{1/2}\left(\sum_{x \in (R/p)} e_p(hx^2/(p\delta))\right)^{-1}$ where p is a prime and $h \in R$ is not divisible by p.

(3) $J_\Xi\left(\begin{pmatrix} p & 0 \\ 0 & 1/p \end{pmatrix}, z\right) = N(p)^{-1/2}$.

A key property of J_Ξ is that it is a partial automorphy factor. To be precise, it has the following properties (see [9]):

(a)$J_\Xi(y_1 x y_2, z) = h(y_1, x y_2(z)) J_\Xi(x, y_2(z)) h(y_2, z)$, if y_1, y_2 belong to $\mathbf{D}$.

(b)$J_\Xi(k^{-1}, z) = J_\Xi(k, k^{-1}(z))^{-1}$, where $k \in \mathbf{D}\sigma\mathbf{D}$ with $\sigma = \begin{pmatrix} 1/a & 0 \\ 0 & a \end{pmatrix}$ with a relatively prime to 2.

Let $\gamma \in \Gamma_{\mathbf{c}}$. We now give a complicated, but still useful, formula for $h(\gamma, z)$.

For $d \in R - \{0\}$, define

$$\epsilon(d) = (i \operatorname{sgn} d)^{1/2} 2^{-n/2} D^{-1/2} \sum_{v \in \delta^{-1}/2R} e(-v^2 d/4).$$

We also define $\widetilde{\epsilon}(d) = i^s$ where s is the number of negative embeddings of d. Then [4, p. 142] tells us that

$$h(\gamma, z) = \epsilon(d_\gamma)\widetilde{\epsilon}(d_\gamma)(\epsilon_{c_\gamma})^*(a_\gamma)(c_\gamma z + d_\gamma)^{1/2} \tag{2.1.1}$$

Also, by [4, p. 146] we have

$$\theta\left(\begin{pmatrix} 0 & -\delta^{-1} \\ \delta & 0 \end{pmatrix} z\right) / \theta(z) = (-\mathbf{i}z)^{1/2} N(\delta)^{1/2}.$$

We extend the notation $h(\gamma, z)$ to this case by defining

$$h\left(\begin{pmatrix} 0 & -\delta^{-1} \\ \delta & 0 \end{pmatrix}, z\right) = (-\mathbf{i}z)^{1/2} N(\delta)^{1/2} \tag{2.1.2}$$

For each totally positive prime element $p \in R$ we define a *Hecke operator* T_{p^2} on $M(\mathbf{c}, \psi)$ that sends f to $f \mid T_{p^2}$ where

$$\begin{aligned} f \mid T_{p^2} = N(p)^{-3/2}\overline{\psi_\infty(p)} & \left(\sum_{b \in R/p^2} f \mid \left[\begin{pmatrix} 1/p & 2b/(\delta p) \\ 0 & p \end{pmatrix}, J_\Xi\left(\begin{pmatrix} 1/p & 2b/(\delta p) \\ 0 & p \end{pmatrix}, z\right)\right] \right. \\ & + \overline{\psi_{\mathbf{c}}(p)} \sum_{h \in (R/p)^\times} f \mid \left[\begin{pmatrix} 1 & 2h/(\delta p) \\ 0 & 1 \end{pmatrix}, J_\Xi\left(\begin{pmatrix} 1 & 2h/\delta p \\ 0 & 1 \end{pmatrix}, z\right)\right] \\ & \left. + \overline{\psi_{\mathbf{c}}(p^2)} f \mid \left[\begin{pmatrix} p & 0 \\ 0 & 1/p \end{pmatrix}, J_\Xi\left(\begin{pmatrix} p & 0 \\ 0 & 1/p \end{pmatrix}, z\right)\right] \right). \end{aligned}$$

2.2. **Action of the Hecke operator on Fourier coefficients.** The next proposition, which is a restatement of ([9], Proposition 5.4) gives the explicit action of T_{p^2} on the Fourier coefficients of a modular form. In particular it also shows that if p and p' are two totally positive elements that generate the same prime ideal, then T_{p^2} coincides with $T_{p'^2}$

Proposition 2.1 (Shimura)**.** *Suppose p be a totally positive prime element of R and $f \in M(\mathbf{c}, \psi)$ is given by*

$$f(z) = \sum_{\xi \in R} a(\xi) e(\xi z/2).$$

Then

$$(f \mid T_{p^2})(z) = \sum_{\xi \in R} b(\xi) e(\xi z/2)$$

where

$$\psi_\infty(p) b(\xi) = a(\xi p^2) + \begin{cases} \overline{\psi_{\mathbf{c}}(p)} N(p)^{-1} (\frac{\xi}{p}) a(\xi) + \overline{\psi_{\mathbf{c}}(p^2)} N(p)^{-1} a(\xi/p^2) & \text{if } p \nmid \mathbf{c} \\ 0 & \text{if } p \mid \mathbf{c} \end{cases}$$

where $a(\xi/p^2) := 0$ if $p^2 \nmid \xi$.

Corollary 2.1. *Suppose*

$$f(z) = \sum_{\xi \in R} a(\xi) e(\xi z/2)$$

is an element of $M(\mathbf{c}, \psi)$ and $f \mid T_{p^2} = c_p f$ for some prime $p \mid \mathbf{c}$. Then

$$a(\xi p^{2n}) = (\psi_\infty(p))^n c_p^n a(\xi)$$

and $\mid c_p \mid \leq 1$.

Proof. The assertion about $a(\xi p^2)$ follows from the above Proposition. Now Corollary 1.1 implies that $\mid c_p \mid \leq 1$. □

3. Easy pickings

3.1. **Eigenforms of Hecke operators.** Consider the Petersson scalar product $< f, g >$ on $M(\mathbf{c}, \psi)$. The definition is analogous to the classical case, see for instance [9].

By standard calculations [9, Proposition 5.3], $\overline{\psi^*(p^2)} T_{p^2}$ is a Hermitian operator if p does not divide $\mathbf{c}$. Hence:

Lemma 3.1. *There is a basis of $M(\mathbf{c}, \psi)$ consisting of eigenforms for all the T_{p^2} where $p >> 0$ is a prime in R and $p \nmid \mathbf{c}$.*

So it is important to study the modular forms that are eigenvalues for the Hecke operators. But first we prove an auxillary lemma.

Lemma 3.2. *The following hold:*

(a) There is a basis of $M(\mathbf{c}, \psi)$ consisting of forms whose coefficients belong to a number field.

(b)If $f(z) = \sum_{\xi \in R} a(\xi) e(\xi z/2) \in M(\mathbf{c}, \psi)$ has each $a(\xi)$ algebraic, then the $a(\xi)$ have bounded denominators (i.e. there exists a non zero integer D such that $Da(\xi)$ is an algebraic integer for all ξ).

Proof. (a) is just [9, Proposition 8.5] while (b) follows easily from Theorem 1.2 above. □

Lemma 3.3. *Let $f(z) = \sum_{\xi \in R} a(\xi)e(\xi z/2) \in M(\mathbf{c}, \psi)$ be an eigenvector of T_{p^2} with eigenvalue c_p where $p \nmid \mathbf{c}$. Suppose $0 << m \in R$ such that $p^2 \nmid m$. Then:*

(a) $a(mp^{2n}) = a(m)\overline{\psi_{\mathbf{c}}(p)^n}(\frac{m}{p})^n$ for every $n \geq 0$

(b) If $a(m) \neq 0$, then $p \nmid m$ and $c_p = \psi^*(p)(\frac{m}{p})(1 + N(p)^{-1})$

Proof. Since T_{p^2} maps forms with algebraic coefficients into themselves, it follows from Lemma 3.2 by simple linear algebra that the eigenvalue c_p is algebraic and that the corresponding eigenspace is generated by forms with algebraic coefficients. So we assume that the coefficients $a(\xi)$ are algebraic.

Consider the power series $A(T) = \sum_{n=0}^{\infty} a(mp^{2n})T^n$

Using Proposition 2.1, we get, by the same argument as in [7, p. 452].

$$A(T) = a(m)\frac{1 - \alpha T}{(1 - \beta T)(1 - \gamma T)}$$

where

$$\alpha = \overline{\psi_{\mathbf{c}}(p)}N(p)^{-1}(\frac{m}{p})$$

and

$$\beta + \gamma = \psi_\infty(p)c_p, \quad \beta\gamma = \overline{\psi_{\mathbf{c}}(p^2)}N(p)^{-1}.$$

This already implies that $a(m) = 0$ implies $a(mp^{2n}) = 0 \forall n$. Hence we may assume that $a(m) \neq 0$ in which case $A(T)$ is a non zero rational function of T. Viewing $A(T)$ as a function over a suitable finite extension of $\mathbb{Q}_p$, we see using Lemma 3.2(b) that $A(T)$ converges in the p-adic unit disk U defined by $\mid T \mid_p < 1$; hence $A(T)$ cannot have a pole in U. However, since $\beta\gamma = \overline{\psi_{\mathbf{c}}(p^2)}N(p)^{-1}$ one of β^{-1}, γ^{-1} belongs to U. Assume it is β^{-1}. Since $A(T)$ is holomorphic we must then have $\alpha = \beta$. So $A(T) = \frac{a(m)}{(1-\gamma T)}$ and so $a(mp^{2n}) = \gamma^n a(m)$. Since $\beta\gamma \neq 0$ we have $\alpha \neq 0$, hence $p \nmid m$. Moreover $\gamma = \beta\gamma/\alpha = \overline{\psi_{\mathbf{c}}(p)}(\frac{m}{p})$. So $a(mp^{2n}) = \gamma^n a(m) = a(m)\overline{\psi_{\mathbf{c}}(p)^n}(\frac{m}{p})^n$. This proves (a) while (b) follows from $c_p = \overline{\psi_\infty(p)}(\alpha + \gamma)$. □

An element $t \in R$ is called squarefree if it is not divisible by the square of a prime element of R.

Theorem 3.1. *Let*

$$f(z) = \sum_{\xi \in R} a(\xi)e(\xi z/2)$$

be a non zero element of $M(\mathbf{c}, \psi)$ and let $\mathbf{c}'$ be an ideal of R such that $\mathbf{c} \mid \mathbf{c}'$. Assume that for all primes $p \nmid \mathbf{c}'$ we have $f \mid T_{p^2} = c_p f$ where $c_p \in \mathbb{C}$. Then there exists a unique(up to multiplication by an unit) totally positive squarefree element $t \in R$ such that $a(\xi) = 0$ unless $\frac{\xi}{t}$ is the square of an element of R. Moreover

(1) $t \mid \mathbf{c}'$

(2) $c_p = \psi^*(p)(\frac{t}{p})(1 + N(p)^{-1})$ *if* $p \nmid \mathbf{c}'$

(3) $a(\xi u^2) = a(\xi)\overline{\psi_{\mathbf{c}}(u)}(\frac{t}{u})$ *if* $(u\ ,\ \mathbf{c}') = 1$

Proof. . Let $\xi, \xi' \in R$ such that $a(\xi), a(\xi') \neq 0$. We first show that ξ'/ξ is a square. Let P be the set of primes p with $p \nmid (\mathbf{c}'\xi\xi')$. If $p \in P$, the previous lemma shows that

$$\overline{\psi_\infty(p)\psi_{\mathbf{c}}(p)}(\frac{\xi}{p})(1 + N(p)^{-1}) = c_p = \overline{\psi_\infty(p)\psi_{\mathbf{c}}(p)}(\frac{\xi'}{p})(1 + N(p)^{-1})$$

Hence

$$(\frac{\xi}{p}) = (\frac{\xi'}{p})$$

for almost all p. But this means that almost all primes split in the extension $F(\sqrt{\xi\xi'})/F$ and hence by a well known result the extension must be trivial, i.e. ξ'/ξ is a square. Write $\xi = tv^2, \xi' = tv'^2$ with t totally positive square free.

This proves the first assertion of the theorem, i.e. the existence of t. Now write $v = p^n u$ with $p \nmid \mathbf{c}'$ and $(p, u) = 1$. So $\xi = tp^{2n}u^2$. Applying the previous lemma to tu^2 we have $a(\xi) = a(tu^2)\overline{\psi_{\mathbf{c}}(p)^n}(\frac{tu^2}{p})^n$. Hence $a(tu^2) \neq 0$ and part (b) of the lemma above shows that $p \nmid t$ and $c_p = \psi^*(\mathfrak{p})(\frac{t}{p})(1 + N(p)^{-1})$.

Hence every prime factor of t divides $\mathbf{c}'$; since t is squarefree this implies $t \mid \mathbf{c}'$, and (1) ,(2) are proved. As for (3), it is enough to check it for $u = p$ with $p \nmid \mathbf{c}'$ and u a unit. The case of $u = p$ follows from writing $\xi = \xi_0 p^{2a}$ with $p^2 \nmid \xi_0$ and applying part (a) of the previous lemma, while the case of an unit follows from [9], Proposition 3.1. □

4. Some operators

Note that all operators on spaces of modular forms defined in this paper act on the *right*. This is done so that composition of operators is compatible with multiplication in $\mathcal{G}$.

Fix a totally positive generator c of $\mathbf{c}$. We define the following operators on $M(\mathbf{c}, \psi)$.

- (The shift operator) For any totally positive $m \in R$, the shift operator $V(m)$ is defined as
$$V(m) = N(m)^{-1/4}\left[\begin{pmatrix} m & 0 \\ 0 & 1\end{pmatrix}, N(m)^{-1/4}\right]$$
Thus
$$(f \mid V(m))(z) = f(mz).$$
- (The symmetry operator) The symmetry operator $W(c)$ is defined as
$$W(c) = [W_0, J_{\Xi}(W_0, z)][V_0(c), N(c)^{-1/4}]$$
where $W_0 = \begin{pmatrix} 0 & -2\delta^{-1} \\ 2^{-1}\delta & 0\end{pmatrix}$ and $V_0(c) = \begin{pmatrix} c & 0 \\ 0 & 1\end{pmatrix}$.
Thus
$$(f \mid W(c))(z) = f \mid \left[\begin{pmatrix} 0 & -2\delta^{-1} \\ 2^{-1}\delta c & 0\end{pmatrix}, (-\mathbf{i}z)^{1/2}N(2^{-2}\delta^2 c)^{1/4}\right]$$
$$= (-\mathbf{i}z)^{-1/2}N(2^{-2}\delta^2 c)^{-1/4}f(\frac{-4}{c\delta^2 z})$$
Observe that $(f \mid W(c)) \mid W(c) = f$.
- (The conjugation operator). The conjugation operator H is defined by
$$(f \mid H)(z) = \overline{f(-\overline{z})}$$

Lemma 4.1. *The operators $V(m), W(c)$ and H take $M(\mathbf{c}, \psi)$ to $M(m\mathbf{c}, \psi\epsilon_m), M(\mathbf{c}, \overline{\psi}\epsilon_c)$ and $M(\mathbf{c}, \overline{\psi})$ respectively. Further, if $f \in M(\mathbf{c}, \psi)$, we have:*

(i) $(f \mid V(m)) \mid T_{p^2} = (f \mid T_{p^2}) \mid V(m)$ *when* $p \nmid m$
(ii) $(f \mid H) \mid T_{p^2} = (f \mid T_{p^2}) \mid H$
(iii) $(f \mid W(c)) \mid T_{p^2} = \psi_{\mathbf{c}}(p^2)(f \mid T_{p^2}) \mid W(c)$ *when* $p \nmid c$

Proof. The statements about H are trivial while those about $V(m)$ follow from ([9], Proposition 3.2) and Proposition 2.1 above.

We now prove the statements concerning $W(c)$. Let $W = \begin{pmatrix} 0 & -2\delta^{-1} \\ 2^{-1}\delta c & 0 \end{pmatrix}$ and $\omega(z) = (-\mathbf{i}z)^{1/2} N(2^{-2}\delta^2 c)^{1/4}$; by definition, $W(c) = [W, \omega(z)]$. Also, recall that $W_0 = \begin{pmatrix} 0 & -2\delta^{-1} \\ 2^{-1}\delta & 0 \end{pmatrix}$.

Let us prove that $W(c)$ takes $M(\mathbf{c}, \psi)$ to $M(\mathbf{c}, \overline{\psi}\epsilon_c)$. We need to show that

$$((f \mid W(c)) \parallel \gamma)(z) = \overline{\psi_{\mathbf{c}}(a_\gamma)}(\epsilon_c)_{\mathbf{c}}(a_\gamma)(f \mid W(c))(z)$$

for any $\gamma \in \Gamma_{\mathbf{c}}$. We write

$$\Gamma = \begin{pmatrix} d_\gamma & -c_\gamma 2^2 \delta^{-2} c^{-1} \\ -\delta^2 2^{-2} c b_\gamma & a_\gamma \end{pmatrix} \in \Gamma_{\mathbf{c}}.$$

Using the fact that $\Gamma^{-1} W \gamma = W$ and $(f \parallel \Gamma) = f$ we are reduced to proving that

$$\frac{h(\Gamma, Wz))}{h(\gamma, z)} = (\epsilon_c)_{\mathbf{c}}(a_\gamma) \frac{(-\mathbf{i}(\gamma z))^{1/2}}{(-\mathbf{i}z)^{1/2}}. \tag{4.0.1}$$

Now put $\gamma' = V_0(c)\gamma V_0(c)^{-1}, z' = cz$. We note that

$$h(\Gamma, Wz) = \theta(W_0 \gamma' z')/\theta(W_0 z').$$

Using the definition $h(G, z) = \theta(Gz)/\theta(z)$ for $G \in \mathbf{D}$ or $G = \begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix} W_0$ we have

$$h(\Gamma, Wz) = \frac{h\left(\begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix} W_0, V_0(1/4)\gamma' z'\right) h(V_0(c/4)\gamma V_0(c/4)^{-1}, cz/4)}{h\left(\begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix} W_0, V_0(1/4) z'\right)}.$$

Use (2.1.2) on the factors $h\left(\begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix} W_0, V_0(1/4)\gamma' z'\right)$, $h\left(\begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix} W_0, V_0(1/4) z'\right)$ to get

$$h(\Gamma, Wz) = \frac{(-\mathbf{i}(\gamma z))^{1/2}}{(-\mathbf{i}z)^{1/2}} h(V_0(c/4)\gamma V_0(c/4)^{-1}, cz/4).$$

Now using (2.1.1) we get

$$h(V_0(c/4)\gamma V_0(c/4)^{-1}, cz/4) = h(\gamma, z)(\epsilon_c)_{\mathbf{c}}(a_\gamma)$$

and this completes the proof of (4.0.1).

As for (iii), it follows from the following identities in $\mathcal{G}$, which can be verified by explicit computation using the (partial) automorphy property of J_Ξ.

(1) Let $B = \begin{pmatrix} 1/p & 2b/(\delta p) \\ 0 & p \end{pmatrix}$ where $b \in (R/p^2)^\times$. Suppose $b' \in (R/p^2)^\times$ be the element such that $bb'c \equiv -1 \bmod p^2$ and let $B' = \begin{pmatrix} 1/p & 2b'/(\delta p) \\ 0 & p \end{pmatrix}$.

Define $\gamma \in \Gamma_{\mathbf{c}}$ by $\gamma = \begin{pmatrix} p^2 & -2b\delta^{-1} \\ -2^{-1}\delta b' c & (1 + bb'c)/p^2 \end{pmatrix}$.

Then we have

$$[\gamma, h(\gamma, z)]\ [B, J_\Xi(B, z)]\ [W, \omega(z)] = [W, \omega(z)]\ [B', J_\Xi(B', z)].$$

(2) Let $C = \begin{pmatrix} 1 & 2h/(\delta p) \\ 0 & 1 \end{pmatrix}$ where $h \in (R/p)^\times$. Suppose $h' \in (R/p)^\times$ be the element such that $hh'c \equiv -1 \bmod p$ and let $B' = \begin{pmatrix} 1/p & 2h'\delta^{-1} \\ 0 & p \end{pmatrix}$.

Define $\gamma, \gamma' \in \Gamma_{\mathbf{c}}$ by $\gamma = \begin{pmatrix} p & -2h\delta^{-1} \\ -2^{-1}\delta h'c & (1+hh'c)/p \end{pmatrix}$, $\gamma' = \begin{pmatrix} (1+hh'c)/p & -2h'\delta^{-1} \\ -2^{-1}\delta hc & p \end{pmatrix}$.

Then we have

$$[\gamma, h(\gamma, z)]\ [C, J_\Xi(C, z)]\ [W, \omega(z)] = [W, \omega(z)]\ [B', J_\Xi(B', z)].$$

$$[\gamma', h(\gamma', z)]\ [W, \omega(z)]\ [C, J_\Xi(C, z)] = (\epsilon_c)_{\mathbf{c}}(p)[B', J_\Xi(B', z)]\ [W, \omega(z)].$$

(3) Let $D = \begin{pmatrix} p & 0 \\ 0 & 1/p \end{pmatrix}$ and $E = \begin{pmatrix} 1/p & 0 \\ 0 & p \end{pmatrix}$.

Then we have

$$[D, N(p)^{-1/2}]\ [W, \omega(z)] = [W, \omega(z)]\ [E, N(p)^{1/2}].$$

$$[W, \omega(z)]\ [D, N(p)^{-1/2}] = [E, N(p)^{1/2}]\ [W, \omega(z)].$$

□

Now, for a totally positive prime $p_0 \in R$ dividing $c/4$ let us write $\Gamma_{\mathbf{c}/p_0}$ as a disjoint union of cosets modulo $\Gamma_{\mathbf{c}}$:

$$\Gamma_{\mathbf{c}/p_0} = \coprod_{\beta \in S} \Gamma_{\mathbf{c}} \beta$$

We define the trace operator $S'(\psi) = S'(\psi, c, p_0)$ on $M(\mathbf{c}, \psi)$ by

$$(f \mid S'(\psi))(z) = \sum_{\beta \in S} \psi(d_\beta)(f \parallel \beta)(z).$$

It is easy to see that this operator does not depend on the choice of the β's. Moreover if $r(\psi) \mid (\mathbf{c}/p_0)$, $S'(\psi)$ takes $M(\mathbf{c}, \psi)$ to $M(\mathbf{c}/p_0, \psi)$ and if $f \in M(\mathbf{c}/p_0, \psi)$ then $f \mid S'(\psi) = uf$ where $u = \mid S \mid$. A routine calculation similar to above also shows that $S'(\psi)$ commutes with T_{p^2} for $p \nmid \mathbf{c}$.

We now define the operator $S(\psi) = S(\psi, c, p_0)$ on $M(\mathbf{c}, \psi)$ by:

$$S(\psi) = \frac{1}{u} N(p_0)^{1/4} W(c) S'(\overline{\psi}\epsilon_c) W(c/p_0)$$

Lemma 4.2. *Let $p_0 \in R$ be a totally positive prime dividing $c/4$, such that $r(\psi\epsilon_{p_0}) \mid (c/p_0)$. Then:*

1. *$S(\psi, c, p_0)$ maps $M(\mathbf{c}, \psi)$ into $M(\mathbf{c}/p_0, \psi\epsilon_{p_0})$*
2. *If m is a totally positive element of R that is prime to p_0, and f belongs to $M(\mathbf{c}, \psi)$, then*
$$f \mid S(\psi, c, p_0) = f \mid S(\psi, mc, p_0).$$
3. *$S(\psi)$ commutes with all the T_{p^2}*
4. *If $g \in M(\mathbf{c}/p_0, \psi\epsilon_{p_0})$, then $(g \mid V(p_0)) \mid S(\psi, c, p_0) = g$.*
5. *Let $p \in R$ be a totally positive prime such that $p \mid (c/4)$, $p \neq p_0$ and $r(\psi\epsilon_p) \mid (c/p)$. If $g \in M(\mathbf{c}/p, \psi\epsilon_p)$, we have:*
$$(g \mid V(p)) \mid S(\psi, c, p_0) = (g \mid S(\psi\epsilon_p, c/p, p_0)) \mid V(p).$$

Proof. The main ingredient for this proof was Lemma 4.1; otherwise the proof is identical to the proof of [6, Lemma 3].

(1) follows directly from Lemma 4.1 and the comments above.

Now note that if $p_0 \nmid m$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ runs over a set of representatives of $\Gamma_{m\mathbf{c}} \backslash \Gamma_{m\mathbf{c}/p_0}$ then $\gamma' = \begin{pmatrix} a & bm \\ c/m & d \end{pmatrix}$ runs over a set of representatives of $\Gamma_{\mathbf{c}} \backslash \Gamma_{\mathbf{c}/p_0}$

To prove (2) we now only need to observe that

$$W(mc)[\gamma, h(\gamma, z)]W(mc/p_0) = [mI, 1]W(c)[\gamma', h(\gamma', z)]W(c/p_0)$$

(3) follows from the commutativity of the Hecke operators with the individual operators that make up $S(\psi)$.

As for (4) observe that

$$(g \mid V(p_0)) \mid W(c) = N(p_0)^{-1/4} g \mid W(c/p_0).$$

The right side is invariant by $\frac{1}{u} S'(\overline{\psi}\epsilon_c)$ and is sent to $N(p_0)^{-1/4} g$ by $W(c/p_0)$.

Finally (5) follows from the following identities which can be checked by explicit computation:

$$\left[\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, N(p)^{-1/4}\right] W(c) = [pI, 1]W(c/p),$$

$$W(c/p_0) = W(c/pp_0)\left[\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, N(p)^{-1/4}\right].$$

.

□

Lemma 4.3. *Let m be a totally positive element of R and $f \in M(\mathbf{c}, \psi)$. Let $g(z) = f(z/m)$. Then $g \parallel \gamma = \psi_{\mathbf{c}}(a_\gamma)\epsilon_m^*(a_\gamma) g$ for all $\gamma \in \Gamma_{\mathbf{c},m}$*

Proof. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix}$ and let $\gamma' = A\gamma A^{-1}$. Note that $\gamma' \in \Gamma_{\mathbf{c}}$.

Then we have

$$[A, N(m)^{1/4}][\gamma, h(\gamma, z)] = [I, \epsilon_m^*(a_\gamma)][\gamma', h(\gamma', z)][A, N(m)^{1/4}].$$

The result is now obtained by letting the above expression act on f. □

Now, for any totally positive prime $p \mid (\mathbf{c}/4)$ we define the operator $U(p) = U(p, \mathbf{c})$ on $M(\mathbf{c}, \psi)$ by:

$$U(p) = N(p)^{-3/4} \sum_{j \in (R/p)} \left[\begin{pmatrix} 1 & 2j\delta^{-1} \\ 0 & p \end{pmatrix}, N(p)^{1/4}\right]$$

Lemma 4.4. *$U(p)$ takes $M(\mathbf{c}, \psi)$ to $M(\mathbf{c}, \psi\epsilon_p)$ and if*

$$f(z) = \sum_{\xi \in R} a(\xi) e(\xi z/2),$$

then

$$(f \mid U(p))(z) = \sum_{\xi \in R} a(p\xi) e(\xi z/2)$$

Proof. Let $A_j = \begin{pmatrix} 1 & 2j\delta^{-1} \\ 0 & 1 \end{pmatrix}$. Put $g(z) = f(z/p)$. Note that

$$f \mid \left[\begin{pmatrix} 1 & 2j\delta^{-1} \\ 0 & p \end{pmatrix}, N(p)^{1/4} \right] = N(p)^{-1/4} g \parallel A_j$$

So for any $\gamma \in \Gamma_c$,

$$f \mid \gamma = N(p)^{-1} \sum_{j \in (R/p)} g \parallel (A_j \gamma).$$

But it is not hard to see that A_j varies over a set of right coset representatives of $\Gamma_{\mathbf{c},p}$ in $\Gamma_{\mathbf{c}}$; hence $A_j\gamma = \gamma_i' A_i$ with $\gamma_i' \in \Gamma_{\mathbf{c},p}$ and distinct j give rise to distinct i. Note also that $a_{\gamma_i'} \equiv a_\gamma \bmod \mathbf{c}$. Therefore

$$\begin{aligned} & N(p)^{-1} \sum_{j \in (R/p)} g \parallel (A_j \gamma) \\ & = N(p)^{-1} (\psi\epsilon_p)_{\mathbf{c}}(a_\gamma) \sum_{j \in (R/p)} g \parallel A_i \\ & = (\psi\epsilon_p)_{\mathbf{c}}(a_\gamma) f \end{aligned}$$

This proves that $U(p)$ takes $M(\mathbf{c}, \psi)$ to $M(\mathbf{c}, \psi\epsilon_p)$.

As for the assertion about the Fourier coefficients, note that

$$\begin{aligned} (f \mid U(p))(z) &= N(p)^{-1} \sum_{j \in (R/p)} f(\frac{z + 2j\delta^{-1}}{p}) \\ &= N(p)^{-1} \sum_{\xi \in R} a(\xi) e(\xi z/2p) \left(\sum_{j \in (R/p)} e(\frac{\xi j \delta^{-1}}{p}) \right). \end{aligned}$$

The result now follows from the fact that

$$\sum_{j \in (R/p)} e(\frac{\xi j \delta^{-1}}{p}) = \begin{cases} N(p) & \text{if } p | \xi \\ 0 & \text{otherwise} \end{cases}$$

□

Finally, define the operator $K(p) = 1 - U(p, p\mathbf{c})V(p)$.

Lemma 4.5. *If $f(z) = \sum_{\xi \in R} a(\xi) e(\xi z/2) \in M(\mathbf{c}, \psi)$ then $f \mid K(p) \in M(\mathbf{c}p^2, \psi)$ and equals $\sum_{(\xi,p)=1} a(\xi) e(\xi z/2)$. Further, if $p' \nmid p\mathbf{c}$ then $T_{p'^2}$ and $K(p)$ commute.*

Proof. . This follows immediately from the above lemma and the properties of $V(m)$ proved earlier. □

5. Newforms

5.1. **Definition of newforms and basic results.** Let $f \in M(\mathbf{c}, \psi)$ be an eigenvector of all but finitely many T_{p^2}. We say that f is an oldform if there exists a totally positive prime p dividing $\mathbf{c}/4$ such that one of the following hold.

(a) $r(\psi)$ divides $(\mathbf{c}/p)$ and $f \in M(\mathbf{c}/p, \psi)$.

(b) $r(\psi\epsilon_p) \mid (\mathbf{c}/p)$ and $f = g \mid V(p)$ with $g \in M(\mathbf{c}/p, \psi\epsilon_p)$.

We denote by $M^O(\mathbf{c},\psi)$ the subspace of $M(\mathbf{c},\psi)$ spanned by oldforms. If $f \in M(\mathbf{c},\psi)$ is an eigenvector of all but finitely many T_{p^2} and f does not belong to $M^O(\mathbf{c},\psi)$, we say that f is a newform of level $\mathbf{c}$.

The following two lemmas are proved exactly as in [6]. They are essentially formal consequences of all the lemmas in the previous subsection.

Lemma 5.1. *The symmetry operator and the conjugation operator take oldforms to oldforms and newforms to newforms.*

Lemma 5.2. *Let $h \in M^0(\mathbf{c},\psi)$ be a non zero eigenform of all but finitely many $T(p^2)$. Then there is a proper divisor $\mathbf{c}'$ of $\mathbf{c}$, a character χ such that $r(\chi) \mid \mathbf{c}'$ and a newform $g \in M(\mathbf{c}',\psi)$ such that h and g have the same eigenvalues for almost all $T(p^2)$.*

We also have

Lemma 5.3. *Let p be a totally positive prime, and let $f(z) = \sum_{\xi\in R} a(\xi)e(\xi z/2) \in M(\mathbf{c},\psi)$ be non-zero and assume that $a(\xi) = 0$ for all ξ not divisible by p. Then p divides $\mathbf{c}/4$, $r(\psi\epsilon_p)$ divides $\mathbf{c}/p$ and $f = g \mid V(p)$ with $g \in M(\mathbf{c}/p,\psi\epsilon_p)$*

Proof. Put $g(z) = f(z/p)$ and let $\mathbf{c}' = \mathbf{c}/p$ if $p \mid \mathbf{c}/4$ and $\mathbf{c}' = \mathbf{c}$ otherwise. By Lemma 4.3 we have

$$g \parallel \gamma = (\psi\epsilon_p)_{p\mathbf{c}}(a_\gamma)g$$

for all $\gamma \in \Gamma_{\mathbf{c}',p}$. Moreover, as g has a Fourier expansion with non-zero coefficients only in places corresponding to elements of R, it follows that the above equation holds for $\gamma = \begin{pmatrix} 1 & 2\delta^{-1} \\ 0 & 1 \end{pmatrix}$. By [9, Lemma 3.4], the equation holds for all $\gamma \in \Gamma_{\mathbf{c}'}$. Since g is non-zero this implies that $r(\psi\epsilon_p)|\mathbf{c}'$ which is possible only if p divides $\mathbf{c}/4$. Thus $\mathbf{c}' = \mathbf{c}/p$ and hence $g \in M(\mathbf{c}/p,\psi\epsilon_p)$.

□

The above lemmas allow us to derive our next theorem, which is the main result that enables us to recognize oldforms. The proof of the theorem is identical to that of Theorem 1 in [6] and will not be given here.

Theorem 5.1. *Let m be a totally positive element of R and $f(z) = \sum_{\xi\in R} a(\xi)e_{\mathbf{a}}(\xi z/2)$ be an element of $M(\mathbf{c},\psi)$ such that $a(\xi) = 0$ for all ξ with $(\xi, m) = 1$. Further assume that f is an eigenform of all but finitely many $T_{p'^2}$. Then $f \in M^0(\mathbf{c},\psi)$.*

5.2. **Structure of newforms.** Suppose $f(z) = \sum_{\xi\in R} a(\xi)e(\xi z/2) \in M(\mathbf{c},\psi)$ is a newform. By theorem 3.1 there is a square free $t \in R$, unique up to multiplication by U^2, such that $a(\xi) = 0$ if ξ/t is not a square.

The proofs of the next four lemmas are again identical to the corresponding lemmas in [6] and are omitted.

Lemma 5.4. *We have $t \in U^2$ and $a(1) \neq 0$.*

Lemma 5.5. *Let $g \in M(\mathbf{c},\psi)$ be an eigenform of all but finitely many $T(p^2)$, with the same eigenvalues as f. Then g is a scalar multiple of f.*

Because of Lemma 5.4, we can divide by $a(1)$ and henceforth assume that f is normalized, i.e. $a(1) = 1$.

Lemma 5.6. *Let $f \in M(\mathbf{c},\psi)$ be a newform. Then f is an eigenform for every $T(p^2)$. Further, if $4p \mid \mathbf{c}$, then the eigenvalue $c_p = 0$.*

Lemma 5.7. *The level $\mathbf{c}$ of the newform f is a square and $f \mid W(c)$ is a multiple of $f \mid H$.*

6. L-series and the proof of the main theorem

6.1. **The L-series.** Let $f(z) = \sum_{\xi \in R} a(\xi)e(\xi z/2)$ be an element of $M(\mathbf{c}, \psi)$. For any ideal I, we define $a(I) = a(\xi)$ where ξ is any totally positive generator of I. Because $a(\xi u^2) = \psi_\infty(u)a(\xi)$ for any unit u by ([9], Prop. 5.4), it follows that if we assume that ψ_∞ is trivial, then $a(I)$ is well-defined. In that case we define the L-series $L(s, f)$ by

$$L(s, f) = \sum \frac{a(I)}{N(I)^s}$$

where the sum is taken over all non-zero ideals of R.

Theorem 6.1. *Suppose $f \in M(\mathbf{c}, \psi)$ where ψ_∞ is trivial and assume that $\mathbf{c}$ is the square of an ideal. Then $L(s, f)$ can be analytically continued to an entire function (with the exception of a simple pole at $s = 1/2$ if f is not a cusp form). Moreover, if*

$$\Lambda(s, f) = (2\pi)^{-ns}\Gamma(s)^n N(\delta)^s N(\mathbf{c})^{(s/2)} L(s, f)$$

then the following relation holds

$$\Lambda(s, f) = \Lambda(1/2 - s, g)$$

where $g = f \mid W(c)$.

Proof. Let

$$f(z) = \sum_{\xi \in R} a(\xi)e(\xi z/2)$$

and

$$g(z) = \sum_{\xi \in R} b(\xi)e(\xi z/2).$$

Also let $\mathbf{c} = (c_1)^2$ where c_1 is totally positive (we can do this because F has narrow class number one).

Put $f_1 = f - a(0)$, $g_1 = g - b(0)$. Recall that F_∞° denotes $\prod_{v \in \infty} F_v^+$ which can be naturally identified with $(\mathbb{R}^+)^n$. Thus there is an action of U^2 on $(\mathbb{R}^+)^n$ and for later purposes it is important to note that this action preserves the norm of an element. Now consider the coset space $(\mathbb{R}^+)^n/U^2$. Define the integral

$$\Phi(s) = \int_{(\mathbb{R}^+)^n/U^2} f_1\left(\frac{2\mathbf{i}y}{c_1\delta}\right) \prod_{j=1}^{n} y_j^s \frac{dy_j}{y_j}. \tag{6.1.1}$$

We first observe that this integral is convergent for all s with $Re(s) > 1/2$. Indeed, by the unit theorem, we may choose the fundamental domain F_∞°/U^2 such that the ratios y_i/y_j are all bounded, and hence all the y_j go to zero or infinity together. As the $y_j \to \infty$ the rapid decay of f_1 assures convergence. As they go to zero, we use the following equation, which follows easily from $g = f \mid W(c)$:

$$f_1\left(\frac{2\mathbf{i}}{c_1\delta y}\right) = \prod y_j^{1/2} g_1\left(\frac{2\mathbf{i}y}{c_1\delta}\right) \quad + b(0)\prod y_j^{1/2} \quad - a(0) \tag{6.1.2}$$

to obtain the same result.

Now, write the right side of (6.1.1) as

$$\begin{aligned}
&\sum_{I\neq 0} a(I)\int_{(\mathbb{R}^+)^n/U^2}\sum_{(\alpha)=I,\alpha>>0} e^{-2\pi tr(\alpha y/(c_1\delta))}\prod_{j=1}^{n} y_j^s\frac{dy_j}{y_j}.\\
&=\sum_{\alpha\in R^+/U^2}\sum_{\epsilon\in U^2} a(\alpha\epsilon)\int_{(\mathbb{R}^+)^n/U^2} e^{-2\pi tr(\alpha\epsilon y/(c_1\delta))}\prod_{j=1}^{n} y_j^s\frac{dy_j}{y_j}.\\
&=\sum_{\alpha\in R^+/U^2} a(\alpha)\int_{(\mathbb{R}^+)^n/U^2} e^{-2\pi tr(\alpha y/(c_1\delta))}\prod_{j=1}^{n} y_j^s\frac{dy_j}{y_j}.\\
&=\sum_{\alpha\in R^+/U^2} a(\alpha)\prod_{j=1}^{n}[(2\pi)^{-s}(c_1^{(j)}\delta^{(j)}/\alpha^{(j)})^s\Gamma(s)]\\
&=(2\pi)^{-ns}\Gamma(s)^n N(\delta)^s N(\mathbf{c})^{s/2}L(s,f)\\
&=\Lambda(s,f).
\end{aligned}$$

On the other hand, we have,

$$\begin{aligned}
&\int_{y\in(\mathbb{R}^+)^n/U^2,N(y)<1} f_1(\frac{2\mathbf{i}y}{c_1\delta})\prod_{j=1}^{n} y_j^s\frac{dy_j}{y_j}\\
&=\int_{y\in(\mathbb{R}^+)^n/U^2,N(y)<1}(f(\frac{2\mathbf{i}y}{c_1\delta})-a_0)\prod_{j=1}^{n} y_j^s\frac{dy_j}{y_j}\\
&=-a_0\int_{y\in(\mathbb{R}^+)^n/U^2,N(y)<1}\prod_{j=1}^{n} y_j^s\frac{dy_j}{y_j}+\int_{y\in(\mathbb{R}^+)^n/U^2,N(y)>1} f(\frac{2\mathbf{i}}{yc_1\delta})\prod_{j=1}^{n} y_j^{-s}\frac{dy_j}{y_j}\\
&=-a_0\int_{y\in(\mathbb{R}^+)^n/U^2,N(y)<1}\prod_{j=1}^{n} y_j^s\frac{dy_j}{y_j}+\int_{y\in(\mathbb{R}^+)^n/U^2,N(y)>1}(g(\frac{2\mathbf{i}y}{c_1\delta})-b_0)\prod_{j=1}^{n} y_j^{1/2-s}\frac{dy_j}{y_j}\\
&\quad+b_0\int_{y\in(\mathbb{R}^+)^n/U^2,N(y)<1}\prod_{j=1}^{n} y_j^{s-1/2}\frac{dy_j}{y_j}\\
&=-\frac{a_0C}{s}-\frac{b_0C}{(1/2-s)}+\int_{y\in(\mathbb{R}^+)^n/U^2,N(y)>1} g_1(\frac{2\mathbf{i}y}{c_1\delta})\prod_{j=1}^{n} y_j^{(1/2-s)}\frac{dy_j}{y_j}.
\end{aligned}$$

for some constant C. Note that in the last step, we have used the Dirichlet unit theorem.

Hence we have shown that

$$\Phi(s)+\frac{a_0C}{s}+\frac{b_0C}{(1/2-s)}=\int_{y\in(\mathbb{R}^+)^n/U^2,N(y)>1}[f_1(\frac{2\mathbf{i}y}{c_1\delta})\prod_{j=1}^{n} y_j^s\frac{dy_j}{y_j}+g_1(\frac{2\mathbf{i}y}{c_1\delta})\prod_{j=1}^{n} y_j^{(1/2-s)}\frac{dy_j}{y_j}] \tag{5}$$

The right side consists of integrals over regions that are bounded away from 0 (by the Dirichlet unit theorem) and hence the rapid decay of f_1 and g_1 near infinity imply that these integrals converge for all s. This proves that $\Phi(s)$ is a meromorphic function with simple poles at $0, 1/2$ if f is not a cusp form. As a corollary, we obtain that $L(s,f)$ can be analytically continued to the entire complex plane (with a simple pole at 1/2 if f is not a cusp form).

To see the functional equation just exchange the roles of f and g in (5).

□

We call a prime ideal $\mathfrak{p}$ of R *non-split* if $\mathfrak{p}$ is the unique prime ideal of R that lies above $\mathfrak{p} \cap \mathbb{Z}$. We call an ideal I of R non-split if all its prime divisors are non-split.

Theorem 6.2. *Suppose* $\mathbf{c}$ *is non-split. Let* f *be a normalized newform in* $M(\mathbf{c}, \psi)$ *with* ψ_∞ *trivial. Then* $\mathbf{c} = 4r(\psi)^2$ *and* $f = \frac{1}{2}\theta_\psi$

Proof. . By Theorem 3.1, Lemma 5.4 and Lemma 5.6, we have the product decomposition

$$L(s,f) = \prod_{\mathfrak{p}|\mathbf{c}} \left(1 - \frac{c_p}{N(\mathfrak{p})^{2s}}\right)^{-1} \prod_{\mathfrak{p}\nmid\mathbf{c}} \left(1 - \frac{\psi^*(\mathfrak{p})}{N(\mathfrak{p})^{2s}}\right)^{-1}$$

Furthermore by Lemma 5.7 and Theorem 6.1 we have

$$(2\pi)^{-ns}\Gamma(s)^n L(s,f) = C_1(2\pi)^{-n(1/2-s)}(\Gamma(1/2-s))^n N(c\delta^2)^{1/2-s} L(1/2-s, Hf)$$

for some constant C_1.

Consider, on the other hand, the function $L(2s, \psi)$ defined by

$$L(2s,\psi) = \frac{\psi^*(I)}{N(I)^{2s}} = \prod_{\mathfrak{p}\nmid r(\psi)} \left(1 - \frac{\psi^*(\mathfrak{p})}{N(\mathfrak{p})^{2s}}\right)^{-1}$$

Then, from ([3], p. 78-79) we know that

$$(2\pi)^{-ns}\Gamma(s)^n L(2s,\psi) = C_2(2\pi)^{-n(1/2-s)} N(4r(\psi)^2\delta^2)^{1/2-s}(\Gamma(1/2-s))^n L(1-2s,\overline{\psi})$$

Dividing these equations we have

$$\prod_{\mathfrak{p}\in S}\left(\frac{1 - c_p N(\mathfrak{p})^{-2s}}{1-\psi^*(\mathfrak{p})N(\mathfrak{p})^{-2s}}\right) = C_3 N(\mathbf{c}/4r(\psi)^2)^{-(1/2-s)} \prod_{\mathfrak{p}\in S}\left(\frac{1-\overline{c_p}N(\mathfrak{p})^{2s-1}}{1-\overline{\psi^*(\mathfrak{p})}N(\mathfrak{p})^{2s-1}}\right)$$

where S is the set of prime ideals $\mathfrak{p}$ for which $c_p \neq \psi^*(\mathfrak{p})$, $\mathfrak{p} \mid \mathbf{c}$.

If, for some $\mathfrak{p} \in S$, we have $\psi^*(\mathfrak{p}) \neq 0$, then the left side of the above equation has an infinity of poles on the line $Re(s) = 0$, only finitely many of which can appear on the right side. This can be seen as follows: if $\mathfrak{p}$ is a prime in S then by assumption, it is the only prime with that norm and so we can find infinitely many s such that $\psi^*(\mathfrak{p})N(\mathfrak{p})^{-2s} = 1$ but none of the expressions $c_{p'}N(\mathfrak{p}')^{-2s}, \overline{c_{p'}}N(\mathfrak{p}')^{2s-1}, \overline{\psi^*(\mathfrak{p}')}N(\mathfrak{p}')^{2s-1}$ equals 1 for any $\mathfrak{p}' \in S, \mathfrak{p}' \neq \mathfrak{p}$.

Hence $\mathfrak{p} \in S$ implies $\psi^*(\mathfrak{p}) = 0$, (in other words $\mathfrak{p} \mid r(\psi)$) and hence $c_p \neq 0$ since $c_p \neq \psi^*(\mathfrak{p})$. But $c_p = 0$ if $4\mathfrak{p} \mid \mathbf{c}$. It follows that either S is empty or consists of the unique prime that lies above 2. Meanwhile, the equation simplifies to

$$\prod_{\mathfrak{p}\in S}(1 - c_p N(\mathfrak{p})^{-2s}) = C_4 N(\mathbf{cm}^2/4r(\psi)^2)^s \prod_{\mathfrak{p}\in S}(1 - c'_p N(\mathfrak{p})^{-2s})$$

where $c'_p = N(\mathfrak{p})/\overline{c_p}$ and $\mathbf{m} = \prod_{\mathfrak{p}\in S}\mathfrak{p}$.

We claim that S is empty. Suppose not, then $S = \{\mathfrak{p}\}$ where $\mathfrak{p}$ is the unique prime above 2. Then, if $c_p \neq c_{p'}$ we can find a zero of the left side of the above identity that is not a zero of the right side. Hence we must have $c_p = c_{p'}$. This implies that $\mid c_p^2 \mid= N(\mathfrak{p})$. But that contradicts Corollary 2.1

Thus S is empty and we have $\mathbf{c} = 4r(\psi)^2$, $L(2s,\psi) = L(s,f)$. This implies that for any non zero ideal I which has a common factor with $\mathbf{c}$ we have that $a(I) = \psi^*(L)$ if

$I = L^2$ for some ideal L coprime to $r(\psi)$ and $a(I) = 0$ in all other cases. This, coupled with Theorem 3.1 shows that f and $\frac{1}{2}\theta_\psi$ have the Fourier coefficients at ξ for all $\xi \neq 0$; hence they also have the same constant coefficient. Thus $f = \frac{1}{2}\theta_\psi$. □

6.2. **Proof of Theorem 1.3.**

Proof. . We break the proof into two parts:

1. *The $\theta_{\psi,t}$ are linearly independent.*

Since t and ψ determine χ, each t occurs as the second entry of at most one (χ, t) in $\Omega(\mathbf{c}, \psi)$. Suppose we have

$$\lambda_1\theta_{\psi_1,t_1} + \lambda_2\theta_{\psi_2,t_2} + ... + \lambda_m\theta_{\psi_m,t_m} = 0$$

with the number of primes in the prime decomposition of t_1 being less than or equal to that for the other t_i and $\lambda_i \neq 0$ for each i. Then the coefficient at place t_1 is $2\lambda_1$ for θ_{ψ_1,t_1} and 0 for the others, thus showing that $\lambda_1 = 0$, a contradiction.

2. *The $\theta_{\psi,t}$ span $M(\mathbf{c}, \psi)$.*

We use induction on the number of (not necessarily distinct)prime factors of $\mathbf{c}$. By Lemma 3.1, it suffices to show that any eigenform f of all the $T_{p^2}, p \nmid \mathbf{c}$ is a linear combination of the $\theta_{\chi,t}$ with $(\chi, t) \in \Omega(\mathbf{c}, \psi)$. If f is a newform, this follows from Theorem 6.2. If not, we may assume f is an oldform. Now we have two cases.

In the first case, $r(\psi)$ divides $\mathbf{c}/p$ and $f \in M(\mathbf{c}/p, \psi)$. Since $\mathbf{c}/p$ is also non-split the induction hypothesis shows that f is a linear combination of the $\theta_{\chi,t}$ with (χ, t) in $\Omega(\mathbf{c}/p, \psi)$ and hence in $\Omega(\mathbf{c}, \psi)$.

In the second case, $r(\psi\epsilon_p)$ divides $\mathbf{c}/p$ and $f = V(p)g$ with $g \in M(\mathbf{c}/p, \psi\epsilon_p)$. Because $\mathbf{c}/p$ is non-split, and $(\psi\epsilon_p)$ is totally even because ψ is, the induction hypothesis shows that g is a linear combination of the $\theta_{\chi,t}$ with $(\chi, t) \in \Omega(\mathbf{c}/p, \psi\epsilon_p)$ and hence f is a linear combination of the $\theta_{\chi,tp}$, with $(\chi, tp) \in \Omega(\mathbf{c}, \psi)$. This completes the proof.

□

6.3. **Examples.** In this section we specialize to the case $F = \mathbb{Q}(\sqrt{2})$, and $\mathbf{c} = (\sqrt{2})^n$. For brevity, let $q = \sqrt{2}$. Note that the ring of integers R is simply $\mathbb{Z}(\sqrt{2})$ and the unit group is $< -1 > \times < 1 + q >$. Note also that F has narrow class number 1, and the prime 2 is ramified in F, which allows us to apply the theorems of the last section with $\mathbf{c} = (q)^n$.

The theorem will apply to any Hecke character ψ of F with ψ_∞ trivial and such that $r(\psi)$ divides q^n. For simplicity we only find the quadratic (of order 2) Hecke characters if this type, and give the explicit bases for each of the corresponding spaces of modular forms. It suffices to find the quadratic Dirichlet characters mod q^n that are trivial on units. For that we need to analyze the structure of the groups $(R/q^n)^\times$.

Proposition 6.1. *Let U_n denote the multiplicative group $(R/q^n)^\times$. Then, if $n \leq 4$, U_n is generated by the units of R and hence there is no nontrivial even Hecke character with conductor dividing q^n. On the other hand, if $n > 4$, the following hold:*

(1) *U_n is the direct sum of the cyclic groups generated by $(1 + q), (-1)$ and $(3 + 4q)$.*
(2) *$(1 + q)$ has order $2^{\lfloor \frac{n}{2} \rfloor}$ while $3 + 4q$ has order $2^{\lfloor \frac{n-3}{2} \rfloor}$ in the group $(R/q^n)^\times$.*
(3) *Let $k = \lfloor \frac{n}{2} \rfloor$, $l = \lfloor \frac{n-3}{2} \rfloor$. Then U_n is isomorphic to $(\mathbb{Z}/2^k\mathbb{Z}) \oplus (\mathbb{Z}/2^l\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$*

Proof. . The case $n \leq 4$ can be checked easily by hand.

For the case $n \geq 5$, first observe that the cardinality of U_n is 2^{n-1}. This follows from the fact that the elements of (R/q^n) can be written as $(a + bq)$ with $a \in (\mathbb{Z}/2^k\mathbb{Z})$ and $b \in (\mathbb{Z}/2^{l+1}\mathbb{Z})$ with k, l as in part (3) of the Proposition, and such an element is invertible iff a is odd.

We first prove (2). The same method is used to calculate the orders of $3 + 4q$ and $1 + q$; the idea is to write $(a + bq)^{2^{k+1}} - 1 = (a + bq)^{2^k} - 1)(a + bq)^{2^k} + 1)$. If for some $n > 2$, q^n *exactly* divides $(a + bq)^{2k} - 1)$ then q^{n+2} exactly divides $(a + bq)^{2^{k+1}} - 1$. Since q^5 exactly divides $(1 + q)^4 - 1$ and q^6 exactly divides $(3 + 4q)^2 - 1$, the result follows.

Now, it is easy to see that the subgroups generated by $(1 + q)$ and $(3 + 4q)$ have trivial intersection, and further, that (-1) does not lie in the subgroup generated by these two elements. Thus (a) follows by comparing cardinalities, and clearly (3) is a direct consequence of (1) and (2).

□

Corollary 6.1. *Let ϕ denote the Hecke character ϵ_u where $u = 2 + q$. Then $r(\phi) = (q^5)$ and ϕ is the unique non-trivial Hecke character with ψ_∞ trivial that is quadratic (of order 2) and whose conductor divides q^n.*

Proof. Observe that any Dirichlet character $\text{mod}(q^n)$ that is trivial on the units of R must be, by the previous proposition, a character on the group generated by $3 + 4q$. Furthermore, if this character is quadratic, it must be either the trivial character or the character that takes value -1 on $(3 + 4q)$; it is not hard to see (since $(3 + 4q)$ is inert in the extension $F(\sqrt{u})/F)$ that this corresponds to the Hecke character ϵ_u. □

This leads us to the following theorem.

Theorem 6.3. *Let $n \geq 5$, $\mathbf{c} = (q^n)$, $u = 2 + q$. Let ϕ denote the Hecke character ϵ_u and $\mathbf{1}$ denote the trivial character. Then:*

(1) *A basis for $M(\mathbf{c}, \mathbf{1})$ comprises of the functions $\{\theta_{\mathbf{1},2^k}, 0 \leq k \leq \lfloor\frac{n-4}{2}\rfloor; \theta_{\phi,2^k u}, 0 \leq k \leq \lfloor\frac{n-15}{2}\rfloor\}$. Thus the dimension of the space $M(\mathbf{c}, \mathbf{1})$ is $(\lfloor\frac{n-2}{2}\rfloor + max\{\lfloor\frac{n-13}{2}\rfloor, 0\})$.*

(2) *A basis for $M(\mathbf{c}, \phi)$ comprises of the functions $\{\theta_{\mathbf{1},2^k u}, 0 \leq k \leq \lfloor\frac{n-5}{2}\rfloor; \theta_{\phi,2^k}, 0 \leq k \leq \lfloor\frac{n-14}{2}\rfloor\}$. Thus the dimension of the space $M(\mathbf{c}, \mathbf{1})$ is $(\lfloor\frac{n-3}{2}\rfloor + max\{\lfloor\frac{n-12}{2}\rfloor, 0\})$.*

Proof. . This follows from Theorem 1.3 and the above Corollary. □

7. Potential applications

In this section we put our work in context by mentioning a couple of potential applications which we hope to take up elsewhere.

7.1. **The Congruence number problem.** An ancient Diophantine problem (the so-called congruence number problem) asks for a good criterion to determine whether an integer is the area of a right angled triangle with rational sides. Such integers are referred to as congruent numbers. This was solved by Tunnell [11]. Tunnell's work begins with the observation that n is congruent if and only if the rank of the elliptic curve $E = y^2 = x^3 - n^2x$ over $\mathbb{Q}$ is non-zero. This is easy to prove by elementary number theory. Now, by the Birch–Swinnerton-Dyer conjecture (one direction of which is known in this case by the work of Coates–Wiles) the above condition is equivalent to the value of the L-function $L(E, s)$ at 1 (the central value) being equal to 0. However, it is not hard to show that $L(E, s)$ equals $L(\phi \otimes \epsilon_n, s)$ where ϕ is the unique normalized newform of weight 2, level 32 and trivial character while ϵ_n is the quadratic character associated to

$\mathbb{Q}(\sqrt{n})$. By work of Waldspurger the value $L(\phi \otimes \epsilon_n, 1)$ is related to the value c_n^2 where c_n is the n'th Fourier coefficient of the weight 3/2 modular form that maps to ϕ under the Shimura correspondence.

Tunnell's main contribution to the problem was to find explicitly the weight 3/2 form above. Using the Serre-Stark theorem, he was able to write this form as a product of an explicit theta-series and a standard weight 1 form. As a result, it was possible to express c_n^2 and consequently the vanishing condition on $L(E, 1)$ in a simple combinatorial form.

One may ask the same question over our totally real number field F. We call an element $\alpha \in R$, F-congruent if there exist positive $X, Y, Z \in F$ such that $X^2 + Y^2 = Z^2$ and $XY = 2\alpha$, with possibly a signature restriction. In the case of real quadratic fields, things work out nicely, though with a slight modification [1]. Thus we hope that one can resolve the congruent number problem over F in a manner similar to what was achieved by Tunnell over $\mathbb{Q}$. One of the crucial points is the construction of an appropriate weight 3/2 Hilbert modular form; we hope to achieve this by using our basis of weight 1/2 forms and multiplying it by a appropriate weight 1 Hilbert modular form.

7.2. **Construction of interesting weight 1 forms.** There are not many explicit examples that illustrate the conjectural correspondence between Galois representations and weight 1 forms. Buhler [2] was able to construct a (classical) modular form of level 800 that corresponds to an icosahedral Galois representation. We believe that our main theorem may be useful in constructing interesting (that is, not a base change and non-dihedral) Hilbert modular forms of a specified level whose L- function matches a (possibly icosahedral) Galois representation of that level.

Given a Hilbert modular form f of weight 1 and two forms g_1, g_2 of weight 1/2, fg_1g_2 is a form of weight 2. This suggests the following procedure. We fix a form F of weight 2 and level N and consider the functions $F/(g_1g_2)$ where (g_1, g_2) varies over pairs of basis forms of weight 1/2, as given by our main theorem. It seems likely that this method will lead to the construction of explicit, interesting examples of weight 1 Hilbert modular forms that correspond to Galois representations.

References

[1] S Achimescu. *Hilbert Modular forms of weight* 1/2. PhD thesis, Caltech, 2004.

[2] Joe Buhler. An icosahedral modular form of weight one. In *Modular functions of one variable, V (Proc. Second Internat. Conf., Univ. Bonn, Bonn, 1976)*, pages 289–294. Lecture Notes in Math., Vol. 601. Springer, Berlin, 1977.

[3] Daniel Bump. *Automorphic forms and representations*, volume 55 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1997.

[4] Paul B. Garrett. *Holomorphic Hilbert modular forms*. The Wadsworth & Brooks/Cole Mathematics Series. Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA, 1990.

[5] Stephen Gelbart and I. I. Piatetski-Shapiro. Distinguished representations and modular forms of half-integral weight. *Invent. Math.*, 59(2):145–188, 1980.

[6] J.-P. Serre and H. M. Stark. Modular forms of weight 1/2. In *Modular functions of one variable, VI (Proc. Second Internat. Conf., Univ. Bonn, Bonn, 1976)*, pages 27–67. Lecture Notes in Math., Vol. 627. Springer, Berlin, 1977.

[7] Goro Shimura. On modular forms of half integral weight. *Ann. of Math. (2)*, 97:440–481, 1973.

[8] Goro Shimura. On Eisenstein series of half-integral weight. *Duke Math. J.*, 52(2):281–314, 1985.

[9] Goro Shimura. On Hilbert modular forms of half-integral weight. *Duke Math. J.*, 55(4):765–838, 1987.

[10] Goro Shimura. *Euler products and Eisenstein series*, volume 93 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1997.

[11] J. B. Tunnell. A classical Diophantine problem and modular forms of weight 3/2. *Invent. Math.*, 72(2):323–334, 1983.

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