Sup norms of Maass forms of powerful level

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- Above, X is **arithmetic** and ϕ_n are eigenfunctions of **Hecke operators**; this allows one to bring in number theory, without which the problem is much harder (and is far from being solved).
- QUE says that in an asymptotic sense ϕ_n does not have large peaks. An even simpler way to quantify this is to consider the sup-norm.

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We will focus on the sup norm question.

The basic problem

Let $X = \Gamma \backslash S$ be the quotient of a (fixed) Riemannian symmetric space S by a (possibly varying) discrete arithmetic group of isometries Γ. Let f be a cuspidal Hecke-Maass form on X with $\|f\|_2 = 1$. Give an upper bound on $\|f\|_\infty$ in terms of the Laplace eigenvalues of f and Γ.

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So we are looking for bounds in the eigenvalue/weight aspect, the level aspect or both (the hybrid aspect)

A very active area

- $\mathrm{GL}_2/\mathbb{Q}$ or indefinite $D^\times/\mathbb{Q}$: Iwaniec-Sarnak (1995), Abbes-Ullmo(1995), Donnelly (2001), Jorgenson-Kramer (2004), Rudnick (2005), Xia (2007), Blomer-Holowinsky (2010), Harcos–Templier (2012, 2013), Templier (2010, 2014, 2015), Das-Sengupta (2013), Kiral (2015), Steiner (2015).
- definite $D^\times$ over totally real number fields: VanderKam (1997), Blomer-Michel (2011, 2013).
- GL_2/K or $D^\times/K$, K number field: Koyama (1995), Blomer–Harcos–Milicevic (2014+), Blomer–Harcos–Maga–Milicevic (2016+).

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Next few results are for **eigenvalue aspect only**.

- $GL(3)/\mathbb{Q}$: Holowinsky–Ricotta–Royer (2014+)
- $\mathrm{PGL}(n)/\mathbb{Q}$: Blomer–Maga (2015), Brumley-Templier (2014+).
- $\mathrm{Sp}(4)/\mathbb{Q}$: Blomer-Pohl (2014+).
- semisimple groups: Marshall (2014+)

And some more papers I haven't mentioned...such as those dealing with **lower bounds** etc.

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Let $B^{\mathrm{Maass}}(\lambda, N, \chi)$ be the set of cuspidal Maass newforms f of level N, character χ, weight 0, Laplace eigenvalue λ, and $\int_{\Gamma_0(N)\backslash\mathbb{H}} |f(z)|^2 dz = 1$, where dz denotes the uniform probability measure on $X = \Gamma_0(N)\backslash\mathbb{H}$.

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Our goal is to give non-trivial bounds for $\|f\|_\infty$ in terms of powers of N (and maybe M) and powers of λ.

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- Why care? Intimate connections with conjectures in geometry, quantum mechanics, **subconvexity problem**, etc.

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- We can also consider the space of holomorphic forms $B^{\mathrm{hol}}(k, N, \chi)$. The only modification required is f should be replaced by $y^{k/2}f$.

Upper bounds for sup-norms for $\mathrm{GL}_2/\mathbb{Q}$

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$\|f\|_\infty \ll$	Due to	Year	Restriction
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Nothing known for non-squarefree levels till 2014!

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Let $f \in B^{Maass}(\lambda, N, \mathbf{1})$. Put $\|f\|_\infty = \sup_{z \in \Gamma_0(N)\backslash\mathbb{H}} |f(z)|$. Then, for any $\epsilon > 0$ we have the bound

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- They key new idea in my result was to replace the cusp at infinity by some cusp of width one. Once this is done, the analogue of the key lemma is valid.
- Also several technical issues related to diophantine analysis. However overall strategy of proof broadly similar to Harcos-Templier.

The main result

I now present the main result of this talk. Assume that N is an integer, χ a character mod N, with conductor M, and $f \in B^{\mathrm{Maass}}(\lambda, N, \chi)$. Put $\|f\|_\infty = \sup_{z \in \Gamma_0(N)\backslash\mathbb{H}} |f(z)|$.

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The main result, **general** version:

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Let N be an integer and χ a character mod N with conductor M. Then for any $f \in B^{Maass}(\lambda, N, \chi)$ we have the bound

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where $N = N_0 N_1$ with N_0 equal to the largest integer such that $N_0^2 | N$ and N_1 equal to the smallest integer such that N divides N_1^2,
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- The corresponding result also holds for holomorphic forms ($\lambda^{5/24} \mapsto k^{1/4}$).

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I will first spend some time about some of the interesting features of this theorem, and then move on to some of the ideas behind the proof.

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  2. (Templier, 2014) If $m = 2n_0$, then $\|f\|_\infty \gg_{\lambda,\epsilon} N^{1/2-\epsilon}$.
  3. (S., 2015+) Large values for $m > 4n_0/3$.
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     Work in progress (Hu, S.) should lead to large values **whenever** $m > n_0$.
- The key point is that in the highly ramified case, the corresponding local Whittaker newforms can have large peaks due to a conspiracy of additive and multiplicative characters. This does not happen when χ is not too highly ramified (e.g., when N is squarefree).

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Why is the upper bound worse for highly ramified χ?

- The upper bound we prove is of the order of $N^{1/4}$ for all $m \leq n_0$ and then the exponent increases linearly, and in the end we end up with a bound of around $N^{1/2}$ when $m = 2n_0$.
- This is not so unexpected!
- One can in fact prove **lower bounds**:
  1. (Trivial) $\|f\|_\infty \gg_{\lambda,\epsilon} 1$.
  2. (Templier, 2014) If $m = 2n_0$, then $\|f\|_\infty \gg_{\lambda,\epsilon} N^{1/2-\epsilon}$.
  3. (S., 2015+) Large values for $m > 4n_0/3$.
     Work in progress (Hu, S.) should lead to large values **whenever** $m > n_0$.
- The key point is that in the highly ramified case, the corresponding local Whittaker newforms can have large peaks due to a conspiracy of additive and multiplicative characters. This does not happen when χ is not too highly ramified (e.g., when N is squarefree).
- The fact that our result gets weaker for $m > n_0$ is therefore quite expected.

Squarefree versus powerful levels

In the squarefree case, one can only show an upper bound of $N^{1/3}$ but in the depth aspect the bound approaches $N^{1/4}$. What's the deal with that?

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- In general, the depth aspect seems to full of interesting phenomena waiting to be explored; coming from the behavior of local vectors in highly ramified representations.

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- The powerful case (in particular the depth aspect) appears to behave in a very different manner from the squarefree level aspect.
- The strong level aspect bounds we get come entirely from **local representation theory** at the ramified primes.
- In general, the depth aspect seems to full of interesting phenomena waiting to be explored; coming from the behavior of local vectors in highly ramified representations.
- Finally, the fact that we get much stronger bounds in the depth aspect than the squarefree level aspect is not *unprecedented*: e.g., there is Milicevic's result on subconvexity, as well as..

Squarefree versus powerful levels

QUE in depth aspect (Nelson-Pitale-S, 2014)

Let $\phi_k \in B^{\mathrm{Maass}}(\lambda_k, p^k, \mathbf{1})$ be a sequence of Hecke-Maass cusp forms with λ_k bounded by some absolute constant. Then, for each Maass form $\phi \in B^{\mathrm{Maass}}(\lambda, 1, \mathbf{1})$,

$$\int_{\mathrm{SL}_2\backslash\mathbb{H}} \phi(z)|\phi_k(z)|^2 dz \ll_\phi N^{-\delta}$$

for some absolute constant $\delta > 1/4$.

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This is much stronger than what we can do in the squarefree case, where we merely can prove a logarithmic rate of convergence.

The idea of proof

In the rest of this talk, I will outline the key ideas involved in proving the depth aspect over squares version, in the case of **non**-highly ramified central character:

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Suppose $N = p^{2n_0}$, $M = p^m$ for some prime p and assume that $m \leq n_0$. Then for $f \in B^{\text{Maass}}(\lambda, N, \chi)$, with $\mathrm{cond}(\chi) = M$, we have the bound

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- The proof is best described in the adelic language, and so that is what I will do.
- Best viewed as a local bound. Relies on a careful analysis of Whittaker newvectors and matrix coefficients in highly ramified representations of p-adic groups. No new inputs related to diophantine analysis or the geometry of numbers are required.
- Provides a flexible adelic framework large parts of which will go through in other cases (number fields, higher rank groups).

The Whittaker expansion

Recall: $N = p^{2n_0}$.

Let ϕ be the automorphic form on $\mathrm{GL}_2(\mathbb{A})$ associated to f. Then

$$\|\phi\|_\infty = \|f\|_\infty.$$

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Let π be the automorphic representation generated by ϕ. So π_p has conductor p^{2n_0}, $\pi_{p'}$ is unramified if $p' \neq p$, and π_∞ is a principal series representation of the form $\chi_1 \boxplus \chi_2$ (where for all $y > 0$, $\chi_1(y) = y^{it}$, $\chi_2(y) = y^{-it}$, with $\lambda = \frac{1}{4} + t^2$).

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For any $g = g_{\mathrm{f}} g_\infty$,

$$\phi(g) = \sum_{q \in \mathbb{Q}_{\neq 0}} W_\phi\left(\begin{bmatrix} q & \\ & 1 \end{bmatrix} g\right).$$

The Whittaker expansion (contd.)

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Fact: If $g_\infty(i) = z$, define $T(g_\infty) = \frac{\lambda^{1/2}}{y}$. Then the sum decays very quickly if $|q| > T(g_\infty)$. Moreover, there is an **integer** $Q(g_{\mathrm{f}})$, depending on g_{f}, such that the sum is supported only on those q whose denominator divides $Q(g_{\mathrm{f}})$.

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$$|\phi(g)| \ll_\epsilon (Q(g_{\mathrm{f}}))^{1/2+\epsilon} \lambda^{1/4} y^{-1/2}.$$

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The key point therefore, is to choose an efficient fundamental domain D inside $\mathrm{GL}_2(\mathbb{A})$, such that $\sup_{g \in D} Q(g_{\mathrm{f}}) \frac{\lambda^{1/2}}{y}$ **is as small as possible**.

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Easy fact: The subset

$$D = M \times \{\left[\begin{smallmatrix} y & x \\ & 1 \end{smallmatrix}\right] : y \geq \sqrt{3}/2\}$$

of $G(\mathbb{A})$ is a generating domain in the sense that the natural map from D to $\mathbb{Z}(\mathbb{A})\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}) / \prod_{p' \neq p} \mathrm{GL}_2(\mathbb{Z}_{p'})$ is a surjection.

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It turns out that the optimal choice is to take $M = \mathrm{GL}_2(\mathbb{Z}_p)\left[\begin{smallmatrix} p^{n_0} & \\ & 1 \end{smallmatrix}\right]$. This choice gives us $Q(g_p) = \sqrt{N}$ for all $g_p \in M$, leading to the bound

$$|\phi(g)| \ll_\epsilon N^{1/4+\epsilon}\lambda^{1/4}y^{-1/2} \ll N^{1/4+\epsilon}\lambda^{1/4}$$

for all $g \in D$!

The local result powering our Whittaker bound

Theorem

Let π be a generic irreducible admissible unitarizable representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ such that the conductor of π equals p^{2n_0} and the conductor of ω_π equals p^m. Assume $m \leq n_0$. Let W_π be the Whittaker newform for π normalized so that $W_\pi(1) = 1$. Then for any $g \in \mathrm{GL}_2(\mathbb{Z}_p)\left[\begin{smallmatrix} p^{n_0} & \\ & 1 \end{smallmatrix}\right]$ the following hold:

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1. (**Support of the Whittaker newform**) If for some $y \in \mathbb{Q}_p^\times$, we have $W_\pi(\left[\begin{smallmatrix} y & \\ & 1 \end{smallmatrix}\right] g) \neq 0$, then $y \in p^{-n_0}\mathbb{Z}_p$.

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2. (**Strong average bounds**) Suppose $b = -n_0 + r$ where $r \geq 0$. Then we have

$$\left(\int_{v \in \mathbb{Z}_p^\times} \left| W_\pi(\left[\begin{matrix} vp^b & \\ & 1 \end{matrix}\right] g) \right|^2 d^\times v \right)^{1/2} \ll q^{-r/4}.$$

Amplification

- As described, the method of Whittaker expansion, leads to the bound

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- This is of the correct strength in the level aspect, but not yet in the eigenvalue aspect.
- To achieve further savings in λ, we use amplification.
- The basic idea behind amplification is to choose nice test functions at each place and use them to write down a trace formula involving a family of automorphic forms containing ϕ. By choosing the test function carefully at the unramified primes, we can ensure that the contribution of ϕ is amplified.

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- This ramified test function (first used by Marshall for this problem in the case of compact quotients, and trivial central character) may be viewed as a ramified version of the classical (unramified) amplifier.
- The proof that this ramified amplifier achieves a level aspect bound of $N^{1/4}$ depends on a technical local theorem about matrix coefficients for highly ramified representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ similar in spirit to what I wrote down for the Whittaker newforms earlier.

Putting everything together, we get the theorem:

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Suppose $N = p^{2n_0}$, $M = p^m$ for some prime p and assume that $m \leq n_0$. Then for $f \in B^{\mathrm{Maass}}(\lambda, N, \chi)$, with $\mathrm{cond}(\chi) = M$, we have the bound

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- All the local results are proved for arbitrary local fields of characteristic 0. So it is likely that the methods can be combined with B–H–M–M to obtain hybrid bound for number fields (Edgar Assing).

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- Amplification did not allow us to improve the N-exponent beyond what we achieved already by the Whittaker expansion method. It would be an interesting and challenging problem to resolve this issue.
- Finally, it would be of interest to develop a more flexible and general local theory, valid for more general groups, that does not rely on a nice newform theory.

Thank you for your attention!