Mass equidistribution for Saito-Kurokawa lifts

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Let X be a compact Riemannian surface of negative curvature, $d\mu$ be the volume form, and f_i traverse a sequence of Laplace eigenfunctions on X such that the Laplace eigenvalues $\lambda_i \to \infty$. For any bounded continuous function ϕ on X, as $i \to \infty$,

$$\frac{\int_X \phi(z)|f_i(z)|^2 d\mu}{\int_X |f_i(z)|^2 d\mu} \to \mathrm{vol}(X)^{-1} \int_X \phi(z) d\mu.$$

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Quantum mechanical interpretation: Eigenfunctions correspond to particles, eigenvalues correspond to their energies.

The classical case: Maass forms

Let $M = \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$. Let f_i traverse a sequence of Hecke–Maass cusp forms on M with Laplace eigenvalues $\lambda_i \to \infty$.

$$\langle f_i, f_i \rangle = \int_M |f_i(z)|^2 d\mu, \quad d\mu := \frac{dxdy}{y^2}.$$

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In this case, one can use the additional structure from arithmetic.

QUE for Hecke–Maass cusp forms (eigenvalue aspect)

For any bounded continuous function ϕ on M, as $i \to \infty$,

$$\frac{1}{\langle f_i, f_i \rangle} \int_M \phi(z)|f_i(z)|^2 d\mu \to \mathrm{vol}(M)^{-1} \int_M \phi(z) d\mu.$$

- This was *proved* by Lindenstrauss (2006) and Soundararajan (2010).
- One of the reasons Lindenstrauss won the Fields medal.

The classical case: a holomorphic analogue

Let $M = \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$. Let f_i traverse a sequence of holomorphic cusp forms of weight k_i such that each f_i is a Hecke eigenform and $k_i \to \infty$.

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A holomorphic analogue of QUE was raised explicitly by Luo– Sarnak.

QUE for holomorphic cusp forms (weight aspect)

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- This was *proved* by Holowinsky and Soundararajan (Annals of Math. 2010).
- A key application: equidistribution of zeroes of Hecke cusp forms (Rudnick).

What about QUE for holomorphic forms on higher rank groups?

- There have been generalizations of Lindenstrauss's work on QUE for Maass forms on higher rank groups.
- Today, I am interested in talking about higher rank generalizations of QUE for **holomorphic** forms.
- Simplest higher rank case: holomorphic Siegel cusp forms of degree n with respect to $\mathrm{Sp}_{2n}(\mathbb{Z})$.
- The method of Holowinsky and Soundararajan basically breaks down in these cases (if $n > 1$).

Holomorphic Siegel cusp forms of degree n

Definition of Sp_{2n}

For a commutative ring R, we denote by $\mathrm{Sp}_{2n}(R)$ the set of $2n \times 2n$ matrices $A \in \mathrm{GL}_{2n}(R)$ satisfying the equation $A^t JA = J$ where $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$.

Definition of $\mathbb{H}_n$

Let $\mathbb{H}_n$ denote the set of complex $n \times n$ matrices Z such that $Z = Z^t$ and $\mathrm{Im}(Z)$ is positive definite.

$\mathbb{H}_n$ is a homogeneous space for $\mathrm{Sp}_{2n}(\mathbb{R})$ under the action

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : Z \mapsto (AZ + B)(CZ + D)^{-1}$$

Holomorphic Siegel cusp forms of degree n

Siegel modular forms

A holomorphic Siegel modular form of degree n, full level and weight k is a holomorphic $\mathbb{C}$-valued function F on $\mathbb{H}_n$ satisfying

$$F(\gamma Z) = \det(CZ + D)^k F(Z),$$

for any $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{2n}(\mathbb{Z})$.

Siegel cusp forms

A holomorphic Siegel modulas form of degree n, full level and weight k is a cusp form if all the degenerate Fourier coefficients of F vanish.

Notation: We use $S_k(\mathrm{Sp}_{2n}(\mathbb{Z}))$ to denote the space of holomorphic Siegel cusp forms of degree n, full level and weight k.

Let $M = \mathrm{Sp}_{2n}(\mathbb{Z})\backslash\mathbb{H}$ for some fixed n. Let $F_i \in S_{k_i}(\mathrm{Sp}_{2n}(\mathbb{Z}))$ such that each F_i is a Hecke eigenform and $k_i \to \infty$.

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$$\langle F_i, F_i \rangle = \int_M |F_i(z)|^2 \det(Y)^{k_i} d\mu, \quad d\mu = dXdY \det(Y)^{-n-1}.$$

QUE conjecture for holomorphic Siegel cusp forms (weight aspect)

For any bounded continuous function ϕ on M, as $i \to \infty$,

$$\frac{1}{\langle F_i, F_i \rangle} \int_M \phi(z)|F_i(z)|^2 \det(Y)^{k_i} d\mu \to \mathrm{vol}(M)^{-1} \int_M \phi(z) d\mu.$$

- This was first raised explicitly by Cogdell and Luo (2008) who also proved that the *average* of the measures over a full Hecke basis (dim $\sim k_i^3$) converges to $d\mu$ over fixed compact sets.

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- The simplest lifts for the Siegel case are the Saito–Kurokawa lifts (for $n = 2$); more generally the Ikeda lifts ($n \geq 2$):

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- QUE for classical holomorphic forms ($n = 1$) was initially proved for Eisenstein series and for dihedral/CM forms.
- The simplest lifts for the Siegel case are the Saito–Kurokawa lifts (for $n = 2$); more generally the Ikeda lifts ($n \geq 2$):
  - Liu (2017) showed that if $\phi = E(Z, 1/2 + it)$ is a degenerate Klingen Eisenstein series and F_i traverses a sequence of Ikeda lifts, then

$$\lim_{i \to \infty} \frac{1}{\langle F_i, F_i \rangle} \int_M E(Z, 1/2 + it) |F_i(Z)|^2 \det(Y)^{k_i} d\mu = 0.$$

  - Katsurada–Kim (2022) showed that if $\phi = E(Z, 1/2 + it)$ is a degenerate Siegel Eisenstein series and F_i traverses a sequence of Ikeda lifts, and $n \geq 4$, then

$$\lim_{i \to \infty} \frac{1}{\langle F_i, F_i \rangle} \int_M E(Z, 1/2 + it) |F_i(Z)|^2 \det(Y)^{k_i} d\mu = 0.$$

How did the proof of Holowinsky–Soundararajan go?

What we need for QUE

Let g equal a Hecke–Maass cusp form or unitary Eisenstein series. Need

$$\lim_{i\to\infty} \frac{1}{\langle f_i, f_i\rangle} \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} g(z)|f_i(z)|^2 y^{k_i} d\mu = 0.$$

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Watson–Ichino formula

$$\left(\frac{1}{\langle f_i, f_i\rangle} \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} g(z)|f_i(z)|^2 y^{k_i} d\mu\right)^2 \approx k_i^{-1} L(1/2, f_i \times f_i \times g).$$

Subconvexity conjecture: $L(1/2, f_i \times f_i \times g) \ll_g k_i^{1-\delta}$.
Conclusion: Subconvexity implies QUE.

Proof of holomorphic QUE by Holowinsky–Soundararajan (contd.)

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- Holowinsky, also working on the problem (independently) proved bounds on “shifted convolution sums”.
- Neither approach gives the complete answer, but if one approach fails, it can be shown that the other succeeds!!
- Holowinsky + Soundararajan = QUE for holomorphic modular forms.

What goes wrong if $n > 1$?

For Hecke eigenforms in $S_k(\mathrm{Sp}_{2n}(\mathbb{Z}))$ that are not classical (i.e., for $n > 1$), there is

- No triple product formula. (Note: Watson–Ichino is a special case of refined GGP. Accidental isomorphisms for a "triple product"). So no clear way to relate the integral to L-values.
- No multiplicativity of Fourier coefficients, so the techniques of sieve-theoretic techniques of Holowinsky for dealing with the shifted convolution sum are no longer available.

Saito–Kurokawa lifts

Today, I want to talk about some work with Jääsaari and Lester where we prove QUE for Saito–Kurokawa (SK) lifts under GRH.

- The SK lifts are Hecke eigenforms in $S_k(\mathrm{Sp}_4(\mathbb{Z}))$ that span a Hecke-invariant subspace (called the SK subspace or the Maass subspace).

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- A Hecke eigenform $F \in S_k(\mathrm{Sp}_4(\mathbb{Z}))$ is a SK lift if and only if there exists $f \in S_{2k-2}(\mathrm{SL}_2(\mathbb{Z}))$ such that $L(s, F) = L(s, f)\zeta(s + \frac{1}{2})\zeta(s - \frac{1}{2})$.

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- *Most forms are non-lifts:* The SK subspace has dimension $\asymp k$ while $\dim(S_k(\mathrm{Sp}_4(\mathbb{Z}))) \asymp k^3$.

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- *Most forms are non-lifts:* The SK subspace has dimension $\asymp k$ while $\dim(S_k(\mathrm{Sp}_4(\mathbb{Z}))) \asymp k^3$.
- A Hecke eigenform $F \in S_k(\mathrm{Sp}_4(\mathbb{Z}))$ is a SK lift if and only if it violates the Ramanujan conjecture for Hecke eigenvalues.

Saito-Kurokawa lifts as theta lifts

Using $\mathrm{PD}^\times \simeq SO(3)$ and $\mathrm{PGSp}_4 \simeq SO(5)$, we have

$$\begin{array}{ccccccccc} \pi & & \mathrm{PD}^\times(\mathbb{A}) & \xleftarrow{\text{Wald}} & \widetilde{\mathrm{SL}}_2(\mathbb{A}) & \xrightarrow{\theta} & \mathrm{PGSp}_4(\mathbb{A}) & & \Pi \\ \uparrow & & & & & & & & \downarrow \\ f & & & & & & & & F \end{array} \tag{1}$$

This allows us to take a classical cusp form f of weight $2k-2$ and produce a SK lift $F \in S_k(\mathrm{Sp}_4(\mathbb{Z}))$.

The main result

Theorem 1 (Jääsaari–Lester–S)

Let $F_i \in S_{k_i}(\mathrm{Sp}_4(\mathbb{Z}))$ be a Hecke eigenform in the Saito–Kurokawa space, with $k_i \to \infty$. Assume the Generalized Riemann Hypothesis. For any bounded continuous function ϕ on $\mathrm{Sp}_4(\mathbb{Z})\backslash\mathbb{H}_2$, as $i \to \infty$, we have

$$\frac{1}{\langle F_i, F_i \rangle} \int_M \phi(z)|F_i(z)|^2 \det(Y)^{k_i} d\mu \to \mathrm{vol}(M)^{-1} \int_M \phi(z) d\mu.$$

In the rest of this talk I will sketch the key ideas in the proof of this theorem.

Fourier expansion of Siegel cusp forms of degree 2

The Fourier expansion

Let $F(Z) \in S_k(\mathrm{Sp}_4(\mathbb{Z}))$. Then we can write

$$F(Z) = \sum_{S \in \Lambda_2^+} a(F,S) e^{2\pi i \mathrm{Tr} SZ}, \quad a(F,S) \in \mathbb{C}.$$

Above, $\Lambda_2^+ := \{S = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} : \qquad a, b, c \in \mathbb{Z},\ S > 0\}$.

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- For $S = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} \in \Lambda_2^+$, let $\mathrm{disc}(S) := b^2 - 4ac < 0$. So for each discriminant $d < 0$, there are exactly $h(d)$ inequivalent Fourier coefficients of discriminant d.

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- *Fourier coefficients contain far more information than Hecke eigenvalues.*

Let $F \in S_k(\mathrm{Sp}_4(\mathbb{Z}))$ that is a SK lift. **Two crucial properties**:

Independence of class group element

The Fourier coefficient $a(F, \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix})$ depends only on $d = b^2 - 4ac$ and $L = \gcd(a, b, c)$. In particular, if d is a fundamental discriminant, all the $h(d)$ inequivalent Fourier coefficients $a(F, S)$ for $\mathrm{disc}(S) = d$ coincide!

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In fact, the Fourier coefficients come from a half-integer weight form.

Explicit relation to half-integer weight forms

There exists a Hecke eigenform $\tilde{f} \in S_{k-\frac{1}{2}}(\Gamma_0(4))$ so that

$$a(F, \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}) = a(\tilde{f}, 4ac - b^2)$$

whenever $\gcd(a, b, c) = 1$.

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Waldspurger's theorem: $|a(F, \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix})|^2 \approx L(1/2, f \times \chi_{b^2-4ac})$.

Key ideas of the proof

- As a starting point we want a collection of incomplete Poincare series (of weight 0) on Sp_4 and hope they span the space of all smooth functions on M.
- One can attach Poincare series to any parabolic of Sp_4. Our first attempt was the minimal parabolic. But this does not work because they only span the subspace of forms with a Whittaker model!

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- As a starting point we want a collection of incomplete Poincare series (of weight 0) on Sp_4 and hope they span the space of all smooth functions on M.
- One can attach Poincare series to any parabolic of Sp_4. Our first attempt was the minimal parabolic. But this does not work because they only span the subspace of forms with a Whittaker model!
- The correct choice is the Siegel parabolic, because its unipotent parabolic is abelian.
- Unfolding gives us a shifted convolution sum of Fourier coefficients of Siegel cusp forms.

Poincare series and unfolding

Let $h \in C_c^\infty(\mathbb{H} \times \mathbb{R}^+)$. Let (ℓ_1, ℓ_2, ℓ_3) be a triple of non-negative integers.

- The data defines a Poincare series $P^h_{\ell_1,\ell_2,\ell_3}(Z) \in L^2(\mathrm{Sp}_4(\mathbb{Z})\backslash\mathbb{H}_2)$.

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- Denote $L = \begin{bmatrix} \ell_1 & \ell_2/2 \\ \ell/2 & \ell_3 \end{bmatrix}$. For any $F \in S_k(\mathrm{Sp}_4(\mathbb{Z}))$, we obtain by unfolding

$$\int_M P^h_{\ell_1,\ell_2,\ell_3}(Z)|F(z)|^2 \det(Y)^k d\mu = \sum_{T \in \Lambda_2^+} a(T)a(T+L)W^h_{\ell_1,\ell_2,\ell_3}(T),$$

  where $W^h_{\ell_1,\ell_2,\ell_3}(T)$ is a "weight" function.
- We have a shifted convolution sum problem with two cases depending on whether (ℓ_1, ℓ_2, ℓ_3) equals 0 or not.

The off-diagonal terms

For $(\ell_1, \ell_2, \ell_3) \neq (0,0,0)$ need to show as $k \to \infty$,

$$\frac{1}{\langle F, F\rangle} \int_M P^h_{\ell_1,\ell_2,\ell_3}(Z)|F(z)|^2 \det(Y)^k d\mu \to 0.$$

Using the unfolding, the relation $a(F, \left[\begin{smallmatrix} a & b/2 \\ b/2 & c \end{smallmatrix}\right]) = a(g, 4ac - b^2)$ and Waldspurger's theorem, we are reduced to showing something like

$$\frac{1}{k^3} \sum_{r,m,n \asymp k} \sqrt{L(\frac{1}{2}, f \times \chi_{r^2-4mn}) L(\frac{1}{2}, f \times \chi_{(r+\ell_1)^2 - 4(m+\ell_2)(n+\ell_3)})} \longrightarrow 0$$

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as $k \to \infty$. Assuming GRH, we prove (a refined version of above) using Soundararajan's method for moments, obtaining a savings of $(\log(k)^{1/4}$.

The diagonal terms

- Instead of working with $P^h_{0,0,0}(Z)$, we do an initial sum on the Levi, then use the spectral decomposition of $L^2(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H})$ and convert the Poincare series to an Eisenstein series!

The diagonal terms

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- The data defines an incomplete Eisenstein series (induced from the Siegel parabolic) $E(Z; g, \kappa)$ on $\mathrm{Sp}_4(\mathbb{Z})\backslash\mathbb{H}_2$.
- We need to prove as $k \to \infty$,

$$\frac{1}{\langle F, F\rangle} \int_M E(Z; g, \kappa)|F(z)|^2 \det(Y)^k d\mu \to 2\langle g, 1\rangle \int_0^\infty \kappa(\lambda)\lambda^{-4} d\lambda.$$

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- We *unfold* the above. This reduces again to a sum involving squares of Fourier coefficients of half-integer weight forms.
- For $g = 1$ (the main term) we use Soundararajan's methods from moments of L-functions to obtain a twisted asymptotic for central L-values, and then combine this result with delicate computations involving the residue of the Rankin–Selberg convolution of the Koecher–Maass series.

The diagonal terms (contd.)

- For g orthogonal to 1, we need

$$\frac{1}{c_k} \sum_{d \asymp k^2} h(d) L(1/2, f \times \chi_d) G(d, g, \kappa) \to 0,$$

$$G(d, g, \kappa) = \frac{|d|^{k-\frac{3}{2}}}{h(d)} \sum_{T \in \mathrm{Cl}(d)} \int_0^\infty \int_{\mathbb{H}} g(z) \lambda^{2k-4} \kappa(\lambda) e^{-4\pi\lambda Tr(T g_z g_z^t)} dz d\lambda.$$

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- Trivially, $G(d,g,\kappa) \ll_{g,\kappa} c_k k^{-3}$; not enough!
- We get (unconditionally) power savings on $G(d,g,\kappa)$ using (essentially) equidistribution of Heegner points

$$G(d,g,\kappa) \ll_{g,\kappa,\epsilon} c_k k^{-3}|d|^{-1/12+\epsilon}$$

which implies what we need.

An application to equidistribution of zero divisors

Theorem 2 (Jääsaari–Lester–S)

Let $F_i \in S_{k_i}(\mathrm{Sp}_4(\mathbb{Z}))$ traverse a sequence of Hecke eigenforms in the Saito–Kurokawa space, with $k_i \to \infty$. Assume GRH. Let $\omega := -\frac{i}{2\pi}\partial\overline{\partial}\log(\det Y)$ be the "canonical" differential form of bidegree $(1,1)$. Fix a smooth compactly supported differential form ϕ of bidegree $(2,2)$ on $\mathrm{Sp}_4(\mathbb{Z})\backslash\mathbb{H}_2$. Let Z_{F_i} denote the zero divisor of F_i. Then

$$\frac{1}{k_i}\int_{Z_{F_i}} \phi \longrightarrow \int_{\mathrm{Sp}_4(\mathbb{Z})\backslash\mathbb{H}_2} \omega \wedge \phi \qquad (2)$$

as $i \longrightarrow \infty$.

Some natural questions

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Thank you!