BOUNDS ON FOURIER COEFFICIENTS AND GLOBAL SUP-NORMS FOR SIEGEL CUSP FORMS OF DEGREE 2

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ABSTRACT. Let F be an L^2-normalized Siegel cusp form for $\mathrm{Sp}_4(\mathbb{Z})$ of weight k that is a Hecke eigenform and not a Saito–Kurokawa lift. Assuming the Generalized Riemann Hypothesis, we prove that its Fourier coefficients satisfy the bound $|a(F, S)| \ll_\epsilon \frac{k^{1/4+\epsilon}(4\pi)^k}{\Gamma(k)} c(S)^{-\frac{1}{2}} \det(S)^{\frac{k-1}{2}+\epsilon}$ where $c(S)$ denotes the gcd of the entries of S, and that its global sup-norm satisfies the bound $\|(\det Y)^{\frac{k}{2}} F\|_\infty \ll_\epsilon k^{\frac{5}{4}+\epsilon}$. The former result depends on new bounds that we establish for the relevant local integrals appearing in the refined global Gan-Gross-Prasad conjecture (which is now a theorem due to Furusawa and Morimoto) for Bessel periods.

1. INTRODUCTION

The problem of bounding the sup-norms of L^2-normalized cuspidal automorphic forms as one or more of their underlying parameters tend to infinity is interesting from several points of view and has been the topic of many recent works, see e.g. [IS95, HT13, Mar14, BP16, Sah17, BHMM20, BT20, Sah20, HS20, KNS22] and the references therein. A basic case of this problem concerns upper-bounds for the global sup-norms of holomorphic cusp forms f of weight k for $\mathrm{SL}_2(\mathbb{Z})$ as $k \to \infty$. Assuming that f is an eigenform, Xia [Xia07] proved the bound

$$\|y^{k/2} f\|_\infty \ll_\epsilon k^{\frac{1}{4}+\epsilon} \|f\|_2. \tag{1}$$

The exponent $1/4$ here is optimal, as one can prove a lower bound of similar strength[1]. The proof uses the Fourier expansion $f(z) = \sum_{n>0} a(f, n) e^{2\pi i n z}$ and crucially relies on Deligne's bound

$$|a(f, n)| \le d(n) n^{\frac{k-1}{2}} |a(f, 1)| \tag{2}$$

for the Fourier coefficients of f as well as the bound $\frac{|a(f,1)|}{\|f\|_2} \ll_\epsilon k^\epsilon$ which follows from the non-existence of Landau–Siegel zeroes [HL94] for the symmetric-square L-function attached to f. Given these deep facts, the deduction of (1) from the Fourier expansion is fairly direct. The fact that the bound (1) is essentially best possible reflects the special behaviour of the exponential function as a degenerate Whittaker function.

In this paper we are interested in a rank 2 analogue of Xia's result. Namely, let $\mathbb{H}_2$ denote the Siegel upper-half space of degree 2 and let $S_k(\Gamma)$ be the space of holomorphic Siegel cusp forms of weight k transforming with respect to the subgroup $\Gamma = \mathrm{Sp}_4(\mathbb{Z}) \subset \mathrm{Sp}_4(\mathbb{R})$. As in the rank 1 case considered by Xia, one may hope to exploit the Fourier expansion and obtain strong bounds on the sup-norm in this setting. Recall that the Fourier expansion of $F \in S_k(\Gamma)$ takes the form

$$F(Z) = \sum_{S \in \Lambda_2} a(F, S) e^{2\pi i \mathrm{Tr}(SZ)}, \qquad Z \in \mathbb{H}_2, \tag{3}$$

where

$$\Lambda_2 = \left\{ \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} : \quad a, b, c \in \mathbb{Z}, \quad a > 0, \quad d := b^2 - 4ac < 0 \right\}.$$

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[1]The lower bound of $k^{\frac{1}{4}-\epsilon}$ is obtained high in the cusp. A variant of the sup-norm problem focusses not on global bounds but on bounds over a fixed compact set Ω where it is an open problem to improve upon the exponent $1/4$. Indeed, the much stronger upper bound $\|y^{k/2} f|_\Omega\|_\infty \ll_\epsilon k^\epsilon$ is expected to hold.

For $S = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} \in \Lambda_2$, we define its discriminant $\operatorname{disc}(S) = -4\det(S) = b^2 - 4ac$ and its content $c(S) = \gcd(a, b, c)$. Note that $c(S)^2$ divides $\operatorname{disc}(S)$. If $d = \operatorname{disc}(S)$ is a fundamental discriminant[2], then S is called *fundamental* in which case clearly $c(S) = 1$.

The Fourier coefficients of Hecke eigenforms in $S_k(\Gamma)$ are mysterious and poorly understood objects. Unlike in the rank 1 case, they contain much more information than just the Hecke eigenvalues and are closely related to central values of L-functions. However, there exist special forms in $S_k(\Gamma)$ known as the *Saito-Kurokawa lifts* for which the Fourier coefficients are relatively better understood. In fact, the Fourier coefficients of a Saito-Kurokawa lift F can be explicitly written in terms of the Fourier coefficients of a classical half-integral weight form $g \in S_{k-\frac{1}{2}}(\Gamma_0(4))$ and there exists a simple relation between the Petersson norms of F and g. Using these facts, Blomer [Blo15, Sec. 4] observed that if a Hecke eigenform F is a Saito–Kurokawa lift, then under the Generalized Lindelöf hypothesis (GLH) one has the following bound[3] on the Fourier coefficients

$$(4) \qquad \frac{|a(F,S)|}{\|F\|_2} \ll_\epsilon \frac{k^{1/4+\epsilon}(4\pi)^k}{\Gamma(k)} c(S)^{\frac{1}{2}} \det(S)^{\frac{k}{2} - \frac{3}{4} + \epsilon}$$

where the Petersson norm $\|F\|_2$ is defined via $\|F\|_2^2 = \langle F, F \rangle = \int_{\Gamma \backslash \mathbb{H}_2} |F(Z)|^2 (\det Y)^{k-3} dX dY$.

Using the bound (4), Blomer obtained under GLH the following essentially optimal bound [Blo15, Theorem 2] on the sup-norm of a Hecke eigenform $F \in S_k(\Gamma)$ that is a Saito–Kurokawa lift:

$$(5) \qquad \|(\det Y)^{\frac{k}{2}} F\|_\infty \ll_\epsilon k^{\frac{3}{4}+\epsilon} \|F\|_2.$$

The main obstacle to generalizing (5) to non-Saito–Kurokawa lifts and obtaining a global sup-norm bound for any Hecke eigenform $F \in S_k(\Gamma)$ lies in obtaining a bound similar to (4) for non-lifts. A key step for fundamental S was taken in [DPSS20] where weighted averages of fundamental Fourier coefficents for such F were related via the refined Gan–Gross–Prasad (GGP) period conjecture for $(\mathrm{SO}_5, \mathrm{SO}_2)$ (which is now a theorem due to Furusawa and Morimoto [FM22]) to values of higher degree L-functions. Using this, a bound under GRH for the fundamental Fourier coefficients was proved in [DPSS20, Prop 3.17]; see also [JLS23, Theorem C] for a related result which saves an additional power of $\log(\det(S))$ but where the implied constant depends on F.

A main achievement of the present paper is to go *beyond fundamental matrices* and obtain a uniform bound for $\frac{|a(F,S)|}{\|F\|_2}$ under GRH for all $S \in \Lambda_2$. To lay the groundwork for our theorem, recall first that $a(F,S) = a(F, {}^tASA)$ for $A \in \mathrm{SL}_2(\mathbb{Z})$, i.e., the Fourier coefficient $a(F,S)$ depends only on the $\mathrm{SL}_2(\mathbb{Z})$-equivalence class of S. Let $D < 0$ be congruent to 0 or 1 mod 4 and let L be a positive integer. The set of $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes of matrices $S \in \Lambda_2$ whose content $c(S)$ equals L and whose discriminant $\operatorname{disc}(S)$ equals $L^2 D$ can be canonically identified with the class group H_D of the imaginary quadratic order of discriminant D. We view the characters Λ of the finite abelian group H_D as Hecke characters of $K^\times \backslash \mathbb{A}_K^\times$ where $K = \mathbb{Q}(\sqrt{D})$. Note that in the special case that D is a fundamental discriminant, these are precisely the characters of the ideal class group. We prove the following theorem.

Theorem 1.1 (see Theorem 3.5). *Let $F \in S_k(\Gamma)$ be a Hecke eigenform with Fourier expansion given by (3). Assume that F is not a Saito–Kurokawa lift and let π be the automorphic representation generated by F. Let $D < 0$ be an integer that is congruent to 0 or 1 mod 4 and let L be a positive*

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[2]Recall that an integer n is a fundamental discriminant if *either* n is a squarefree integer congruent to 1 modulo 4 *or* $n = 4m$ where m is a squarefree integer congruent to 2 or 3 modulo 4.

[3]See also [PS22, Theorem 5.1.6] for an extension to the case of Saito–Kurokawa lifts with square-free level, where the dependence of the implied constant on F is not made explicit.

integer. Then

$$(6) \qquad \sum_{\substack{S \in \Lambda_2/\mathrm{SL}_2(\mathbb{Z}) \\ c(S)=L, \ \mathrm{disc}(S)=L^2 D}} |a(F,S)|^2 \ll_\epsilon \langle F,F \rangle \frac{(4\pi)^{2k}}{\Gamma(2k-1)} L^{2k-3+\epsilon}|D|^{k-\frac{3}{2}+\epsilon} \sum_{\Lambda \in \widehat{H_D}} \frac{L(1/2, \pi \times \mathcal{AI}(\Lambda))}{L(1,\pi,\mathrm{Ad})}.$$

In the above theorem, we note that the L-values are known to be non-negative [Lap03, Theorem 1.1] and that the length of the sum on each side is equal to $|H_D| \asymp |D|^{\frac{1}{2}+o(1)}$. Under GRH[4], we can bound $\frac{L(1/2,\pi \times \mathcal{AI}(\Lambda))}{L(1,\pi,\mathrm{Ad})} \ll_\epsilon (kD)^\epsilon$. Recalling that $4\det(S) = |\mathrm{disc}(S)| = L^2|D|$ and using the duplication formula for the Gamma function, we obtain under GRH the strong bound (Corollary 3.7)

$$(7) \qquad \sum_{\substack{S \in \Lambda_2/\mathrm{SL}_2(\mathbb{Z}) \\ c(S)=L, \ \mathrm{disc}(S)=L^2 D}} \frac{|a(F,S)|^2}{\|F\|_2^2} \ll_\epsilon \frac{k^{1/2+\epsilon}(2\pi)^{2k}}{\Gamma(k)^2} L^{-1} |L^2 D|^{k-1+\epsilon}.$$

The bound (7) may be viewed as an extension of the bound (4) to non-Saito–Kurokawa lifts[5]. We believe that (7) is optimal as far as the exponents on the right side are concerned. Assuming that most summands on the left side of (7) are of comparable size, one is led to the optimistic and far-reaching conjecture $\frac{|a(F,S)|}{\|F\|_2} \ll_\epsilon \frac{k^{1/4+\epsilon}(4\pi)^k}{\Gamma(k)} \det(S)^{\frac{k}{2}-\frac{3}{4}+\epsilon}$ for individual Fourier coefficients, which refines the famous open conjecture of Resnikoff and Saldaña [RS74] (and whose proof seems well beyond reach even if one were to assume standard conjectures like GRH). On the other hand, dropping all but one term from (7), we obtain the following corollary.

Corollary 1.2. *Assume GRH. For a Hecke eigenform $F \in S_k(\Gamma)$ that is not a Saito–Kurokawa lift, we have for any $S \in \Lambda_2$ the bound*

$$(8) \qquad \frac{|a(F,S)|}{\|F\|_2} \ll_\epsilon \frac{k^{1/4+\epsilon}(4\pi)^k}{\Gamma(k)} c(S)^{-\frac{1}{2}} \det(S)^{\frac{k-1}{2}+\epsilon}.$$

In contrast to (7), the bound (8) is not expected to be optimal (even though it assumes GRH) because we potentially lose a factor of $\frac{\det(S)^{\frac{1}{4}}}{c(S)^{\frac{1}{2}}}$ when we drop all the other terms. In the body of the paper we do not assume GRH but instead assume that $\frac{L(1/2,\pi \times \mathcal{AI}(\Lambda))}{L(1,\pi,\mathrm{Ad})}$ is bounded by a specific power of the analytic conductor and we write down analogous bounds to (7), (8) under this assumption; see Corollary (3.7).

Using our bound (8), we obtain a global sup-norm bound for non Saito–Kurokawa lifts $F \in S_k(\Gamma)$ under GRH.

Theorem 1.3 (see Theorem 3.12). *Assume GRH. Let $F \in S_k(\Gamma)$ be a Hecke eigenform that is not a Saito–Kurokawa lift. Then*

$$\|(\det Y)^{\frac{k}{2}} F\|_\infty \ll_\epsilon k^{\frac{5}{4}+\epsilon} \|F\|_2.$$

The reason the exponent $5/4$ in the above Theorem is weaker than the exponent $3/4$ proved by Blomer for Saito–Kurokawa lifts is the non-optimality of the bound (8). (We expect the true exponent for the sup-norm to be $3/4$ for non-lifts[6] as well).

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[4]Strictly speaking, all we need is GLH and a sufficiently strong zero-free region for $L(s,\pi,\mathrm{Ad})$.

[5]A key point here is that for Saito–Kurokawa lifts F, $a(F,S)$ depends only on $c(S)$ and $\det(S)$, and so for the corresponding sum in the case of the Saito–Kurokawa lifts all terms on the left side are equal. This is not true for non-Saito–Kurokawa lifts.

[6]This is a special case of Conjecture 1.1 of [DK22] and is supported by heuristics of the Bergman kernel as well as the fact that one can prove a lower bound of $k^{3/4-\epsilon}$ for many non-lifts using a method similar to [Blo15, Theorem 1].

The proof of Theorem 1.3 follows from (8) in a relatively straightforward manner via the Fourier expansion, similar to the analysis in [Blo15]. We remark here that (as is often the case) the Fourier expansion gives better results near the cusp. Indeed, our proof shows that if Y is Minkowski-reduced, then we have under GRH the stronger bound $(\det Y)^{\frac{k}{2}}|F(Z)| \ll_\epsilon (\det Y)^{-\frac{1}{4}} k^{\frac{5}{4}+\epsilon}\|F\|_2$. In particular, when Z is "high in the cusp", this provides extra power savings in k. Thus, one may hope to obtain improved sup-norm bounds if one can tackle the "bulk" by different methods.

While we have focussed on the Fourier expansion approach towards the global sup-norm in this paper, alternative approaches toward Theorem 1.3 exist. For example, using a Bergman Kernel approach, Das–Krishna [DK22] have proved the bound $\|(\det Y)^{\frac{k}{2}}F\|_\infty \ll_\epsilon k^{\frac{9}{4}+\epsilon}\|F\|_2$. There is also an exciting new approach to the sup-norm problem via 4th moments and theta kernels introduced by Steiner et al [Ste20, KS22, KNS22] which, if implemented for Siegel cusp forms, could potentially lead to strong bounds.

We end this introduction with a few words about the proof of Theorem 1.1, which should be viewed as the central result of this paper. The reader may be tempted to try to derive Theorem 1.1 (or the stronger Theorem 3.5) from its known special case [DPSS20] for D a fundamental discriminant, by using the Hecke relations between fundamental and non-fundamental coefficients. However, this approach appears not to work. To see why, suppose that we know the values of all the fundamental Fourier coefficients and all the Hecke eigenvalues of F. Using the Hecke relations which can be expressed compactly via Sugano's formula as in [KST12, Theorem 2.10] we can then evaluate sums like $\sum_{\substack{S\in\Lambda_2/\mathrm{SL}_2(\mathbb{Z}) \\ \pi^{-1}(L^{-1}S)=c}} a(F,S)$ with $c(S)=L$, $\mathrm{disc}(S)=L^2 D$ and c a fixed class in H_d where d is the fundamental discriminant attached to D, and $\pi: H_d \to H_D$ is the natural map. However, it does not seem possible to tell apart the individual summands above, i.e., this method cannot separate two non-equivalent, non-fundamental coefficients $a(F,S_1)$ and $a(F,S_2)$ of equal content L in the case that $L^{-1}S_1$ and $L^{-1}S_2$ are images of the *same* fundamental coefficient class under the natural map from $H_d \to H_D$.

Therefore we develop a different method for proving Theorem 1.1. We build upon the explicit refinement of Böcherer's conjecture introduced in [DPSS20] and incorporate characters Λ of H_D which in general correspond to *ramified* Hecke characters of $K = \mathbb{Q}(\sqrt{d})$. (In [DPSS20] we had restricted ourselves to ideal class group characters, i.e., characters of H_d; these correspond to unramified Hecke characters of K.) Using the refined GGP conjecture proved in [FM22], we write the left hand side of (6) as a deformation of the right hand side, where each summand is multiplied by a product over primes $p|D$ of certain local quantities related to local Bessel models for ramified characters. Bounding these purely local quantities constitute the technical heart of this paper. This may be viewed as a *depth aspect* bound for the local Bessel functions as the conductor of Λ becomes large at the primes dividing D. For the main local statements, we refer the reader to Propositions 2.1 and 2.2. In contrast to the similar local quantities for unramified characters computed in [DPSS20], the local quantities associated to ramified characters are not easy to compute explicitly, so we bound them via a soft approach where we exploit the volume of the support of the integrals. We refer the reader to Section 2.4 for details of the argument. It would be of interest to compute these integrals exactly, as this would allow us to replace (6) by an exact identity.

Notation. We use the notation $A \ll_{x,y,\dots} B$ to signify that there exists a positive constant C, depending at most upon x,y,z, so that $|A| \le C|B|$. If the subscripts $x,y,\dots$ are omitted, it means the constant is absolute. We write $A \asymp B$ to mean $A \ll B \ll A$. The symbol ε will denote a small positive quantity.

We say that an integer d is a fundamental discriminant if $\mathbb{Q}(\sqrt{d})$ is a quadratic field whose discriminant is equal to d. For a fundamental discriminant d, we let χ_d be the associated quadratic Dirichlet character. We use $\mathbb{A}$ to denote the ring of adeles over $\mathbb{Q}$ and for a number field F we let

$\mathbb{A}_F$ denote the ring of adeles over F. All L-functions in this paper will denote the finite part of the L-function (i.e., without the archimedean factors), so that for an automorphic representation π of $\mathrm{GL}_n(\mathbb{A})$, we have $L(s, \pi) = \prod_{p < \infty} L(s, \pi_p)$. All L-functions will be normalized to take $s \mapsto 1 - s$. For a finite set of places S we denote $L^S(s, \pi) = \prod_{p \notin S} L(s, \pi_p)$.

For a commutative ring R, we define

$$\mathrm{GSp}_4(R) = \{g \in \mathrm{GL}_4(R) : {}^t g J g = \mu(g) J, \ \mu(g) \in R^\times\}, \qquad J = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix}$$

Here, μ is called the similitude character. Let $\mathrm{Sp}_4(R) = \{g \in \mathrm{GSp}_4(R) : \mu(g) = 1\}$. We occasionally use G to denote GSp_4.

We let $M_2(R)$ denote the ring of 2 by 2 matrices over R, and let $M_2^{\mathrm{sym}}(R)$ be the additive subgroup of symmetric matrices.

Over the real numbers, we have the identity component $G(\mathbb{R})^+ := \{g \in \mathrm{GSp}_4(\mathbb{R}) : \mu(g) > 0\}$. Let $\mathbb{H}_2$ be the Siegel upper half space of degree 2, i.e., the space $\mathbb{H}_2$ consists of the symmetric, complex 2×2-matrices with positive definite imaginary parts. The group $G(\mathbb{R})^+$ acts on $\mathbb{H}_2$ via $g\langle Z \rangle = (AZ + B)(CZ + D)^{-1}$ for $g = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$. We define $j(g, Z) = \det(CZ + D)$ for $g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in G(\mathbb{R})^+$.

2. LOCAL CALCULATIONS

In this section, which is purely local, we will study the *Bessel function* and *Bessel integral* associated to a spherical vector in an irreducible, tempered, principal series representation of $\mathrm{GSp}_4(F)$ where F is a non-archimedean local field of characteristic zero. Our main local results, Proposition 2.1 and Proposition 2.2, quantify the growth of these quantities along diagonal matrices.

2.1. **Basic facts and definitions.**

2.1.1. *Preliminaries.* Throughout Section 2, F will be a non-archimedean local field of characteristic zero. Let $\mathfrak{o}$ be the ring of integers of F, with maximal ideal $\mathfrak{p}$ and uniformizer ϖ. Let $\mathbf{k} = \mathfrak{o}/\mathfrak{p}$ be the residue class field, and q its cardinality. For $x \in F$, let $v(x)$ be the normalized valuation and let $|x| = q^{-v(x)}$ denote the normalized absolute value of x, so that $v(\varpi) = 1$, $|\varpi| = q^{-1}$. Let ψ be a character of F which is trivial on $\mathfrak{o}$ but non-trivial on $\mathfrak{p}^{-1}$.

We use the Haar measure dx on F that assigns $\mathfrak{o}$ volume 1, and we use the Haar measure $d^\times x$ on $F^\times$ that assigns $\mathfrak{o}^\times$ volume 1. So we have $d^\times x = (1 - q^{-1})^{-1} \frac{dx}{|x|}$.

2.1.2. *The Bessel subgroup.* Following [Fur93] and [PS09] we introduce the following notations. Let $a, b, c \in F$ such that $d := b^2 - 4ac \neq 0$. Let

$$(9) \qquad S = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}, \quad \Delta = \begin{bmatrix} b/2 & c \\ -a & -b/2 \end{bmatrix}$$

and note that $d = \mathrm{disc}(S) = -4\det(S)$ and $\Delta^2 = d/4$. If d is not a square in $F^\times$, then let $K = F(\sqrt{d})$ and note that K is isomorphic to $F(\Delta)$ via

$$(10) \qquad F(\Delta) \xrightarrow{\sim} K, \qquad x + y\Delta \longmapsto x + y\frac{\sqrt{d}}{2}.$$

If d is a square in $F^\times$, then let $K = F \oplus F$ and note that K is isomorphic to $F(\Delta)$ via

$$(11) \qquad F(\Delta) \xrightarrow{\sim} K, \qquad x + y\Delta \longmapsto \left(x + y\frac{\sqrt{d}}{2}, x - y\frac{\sqrt{d}}{2}\right).$$

We define

(12) $$T(F) = \{g \in \mathrm{GL}_2(F) : {}^t g S g = \det(g) S\}.$$

One can check that $T(F) = F(\Delta)^\times$, so that $T(F) \cong K^\times$ via the isomorphisms (10), (11) above. We define the Legendre symbol as

(13) $$\left(\frac{K}{\mathfrak{p}}\right) = \begin{cases} -1 & \text{if } K/F \text{ is an unramified field extension,} \\ 0 & \text{if } K/F \text{ is a ramified field extension,} \\ 1 & \text{if } K = F \oplus F. \end{cases}$$

These three cases are referred to as the *inert case*, *ramified case*, and *split case*, respectively. If K is a field, then let $\mathfrak{o}_K$ be its ring of integers and $\mathfrak{p}_K$ be the maximal ideal of $\mathfrak{o}_K$. If $K = F \oplus F$, then let $\mathfrak{o}_K = \mathfrak{o} \oplus \mathfrak{o}$.

Throughout, we will make the following *standard assumptions* (see Section 1 of [PS14]),

(14)
- $a, b \in \mathfrak{o}$ and $c \in \mathfrak{o}^\times$.
- If $d \notin F^{\times 2}$, then d is a generator of the discriminant of K/F.
- If $d \in F^{\times 2}$, then $d \in \mathfrak{o}^\times$.

Under these assumptions, the group $T(\mathfrak{o}) := T(F) \cap \mathrm{GL}_2(\mathfrak{o})$ is isomorphic to $\mathfrak{o}_K^\times$ via the isomorphism $T(F) \cong K^\times$. Note also that these assumptions imply that if we are in the split case ($d \in F^{\times 2}$) then $\frac{b \pm \sqrt{d}}{2} \in \mathfrak{o}$.

We consider $T(F)$ a subgroup of $\mathrm{GSp}_4(F)$ via

(15) $$T(F) \ni g \longmapsto \begin{bmatrix} g & \\ & \det(g)\,{}^t g^{-1} \end{bmatrix} \in \mathrm{GSp}_4(F).$$

Let $N(F)$ be the unipotent radical of the Siegel parabolic subgroup, i.e.,

$$N(F) = \left\{ \begin{bmatrix} 1_2 & X \\ & 1_2 \end{bmatrix} \in \mathrm{GSp}_4(F) : {}^t X = X \right\}$$

and let $R(F) = T(F)N(F)$. We call $R(F)$ the *Bessel subgroup* of $\mathrm{GSp}_4(F)$.

2.1.3. *Bessel models.* For S as in (9), we define a character θ of $N(F)$ by

(16) $$\theta\left(\begin{bmatrix} 1 & & x & y \\ & 1 & y & z \\ & & 1 & \\ & & & 1 \end{bmatrix}\right) = \psi(ax + by + cz) = \psi(\mathrm{tr}(S\begin{bmatrix} x & y \\ y & z \end{bmatrix}))$$

for $x, y, z \in F$. It is easily verified that $\theta(tnt^{-1}) = \theta(n)$ for $n \in N(F)$ and $t \in T(F)$.

Let Λ be any character of $K^\times$ such that $\Lambda|_{F^\times} = 1$. We identify Λ with a character of $T(F)$ using the isomorphism $T(F) \simeq K^\times$. The map $tu \mapsto \Lambda(t)\theta(u)$ defines a character of $R(F)$. We denote this character by $\Lambda \otimes \theta$. Let $\mathcal{S}(\Lambda, \theta)$ be the space of all locally constant functions $B : \mathrm{GSp}_4(F) \to \mathbb{C}$ with the *Bessel transformation property*

(17) $$B(rg) = (\Lambda \otimes \theta)(r)B(g) \qquad \text{for all } r \in R(F) \text{ and } g \in \mathrm{GSp}_4(F).$$

Consider an irreducible, admissible, unitarizable, tempered representation (π, V_π) of trivial central character. If (π, V) is isomorphic to a subrepresentation of $\mathcal{S}(\Lambda, \theta)$, then this realization of π is called a (Λ, θ)-*Bessel model*. It is known that such a model, if it exists, is unique; we denote it by $\mathcal{B}_{\Lambda,\theta}(\pi)$. Since π is unitary, let $\langle\,,\,\rangle$ denote a $\mathrm{GSp}_4(F)$-invariant inner product (unique up to scaling) on V_π. For a vector $v \in V_\pi$, the normalized matrix coefficient attached to v is the function

(18) $$\Phi_v(g) = \frac{\langle \pi(g)v, v \rangle}{\langle v, v \rangle}.$$

Define

$$(19) \qquad J_{\Lambda,\theta}(v) := \int\limits_{F^\times \backslash T(F)} \int\limits_{N(F)}^{\mathrm{st}} \Phi_v(tn)\Lambda^{-1}(t)\theta^{-1}(n)\, dn\, dt,$$

where $\int\limits_{N(F)}^{\mathrm{st}} := \lim_{k\to\infty} \int\limits_{N(\mathfrak{p}^{-k})}$ denotes the stable integral [Liu16, Sect. 3.1]. It can be shown that

the representation π has a (Λ,θ)-Bessel model (i.e., $\mathcal{B}_{\Lambda,\theta}(\pi)$ exists) *if and only if* there is a non-zero vector v in the space of π such that $J_{\Lambda,\theta}(v) \neq 0$, in which case v is said to be a (Λ,θ)-test vector for π. We will refer to $J_{\Lambda,\theta}(v)$ as the *local Bessel integral* (of type (Λ,θ)) associated to v.

Suppose that the representation π has a (Λ,θ)-Bessel model. Fix a realization of π in $\mathcal{B}_{\Lambda,\theta}(\pi)$ and for each vector $v \in V_\pi$ let $B_v \in \mathcal{B}_{\Lambda,\theta}(\pi)$ denote its image. From uniqueness arguments it is easy to show that there exists a non-zero c that depends on Λ, θ, π and on the choice of realization such that for all $v \in V_\pi$ and all $g \in \mathrm{GSp}_4(F)$ we have $|B_v(g)|^2 = c\, J_{\Lambda,\theta}(\pi(g)v)$.

2.1.4. *Subgroups and characters of $T(\mathfrak{o})$.* Recall that

$$T(\mathfrak{o}) := T(F) \cap \mathrm{GL}_2(\mathfrak{o}),$$

and that $T(\mathfrak{o})$ is isomorphic to $\mathfrak{o}_K^\times$ via the isomorphisms (10), (11). We define the subgroup $U_T(m) \subset T(\mathfrak{o})$ via

$$U_T(0) = T(\mathfrak{o}),$$

and for $m \geq 1$,

$$U_T(m) = \{g \in T(\mathfrak{o}) : g = \begin{bmatrix} \lambda & \\ & \lambda \end{bmatrix} \bmod \mathfrak{p}^m, \text{ for some } \lambda \in \mathfrak{o}^\times\}.$$

Let

$$\Delta_0 := \begin{bmatrix} 0 & c \\ -a & -b \end{bmatrix}$$

so that Δ_0 corresponds (under (10), (11) respectively) to the element $\delta_0 := \frac{-b+\sqrt{d}}{2}$ if K is a field, and the element $\delta_0 := \left(\frac{-b+\sqrt{d}}{2}, \frac{-b-\sqrt{d}}{2}\right)$ if $K = F \oplus F$. One can show (using [PS09, Sect. 3]) that

$$(20) \qquad \mathfrak{o}_K = \{x + y\delta_0 : x \in \mathfrak{o}, y \in \mathfrak{o}\}$$

and for $m \geq 1$,

$$U_T(m) = \{x + y\Delta_0 : x \in \mathfrak{o}^\times, y \in \mathfrak{p}^m\}.$$

Using the above description, it is easy to see that (10), (11) induce isomorphisms

$$U_T(m) \xrightarrow{\sim} \mathfrak{o}^\times (1 + \mathfrak{p}^m \mathfrak{o}_K)$$

for each $m \geq 1$.

Any character Λ of $K^\times$ satisfying $\Lambda|_{F^\times} = 1$ can be identified with a character of $F^\times \backslash T(F)$. From the above description, it follows that such a character must be trivial on $U_T(m) \simeq \mathfrak{o}^\times(1 + \mathfrak{p}^m \mathfrak{o}_K)$ for some m. We define

$$c(\Lambda) = \min\{m \geq 0 : \Lambda|_{U_T(m)} = 1\}.$$

2.2. **Main results.** For the rest of Section 2, let π be a tempered, spherical, irreducible principal series representation of $\mathrm{GSp}_4(F)$, i.e., π is a tempered representation of Type I in the notation of [RS07]. We also assume throughout that π is of trivial central character. In particular, π is the unramified constituent of a representation $\chi_1 \times \chi_2 \rtimes \sigma$ induced from a character of the Borel subgroup associated to unramified characters χ_1, χ_2, σ of $F^\times$ satisfying $\chi_1\chi_2\sigma^2 = 1$. We put

$$\alpha = \sigma(\varpi), \quad \beta\sigma(\varpi)\chi_1(\varpi).$$

Note that the temperedness of π implies that χ_i and σ are unitary and therefore $|\alpha| = |\beta| = 1$. Let ϕ be a (unique up to multiples) spherical vector in V_π, i.e., ϕ is fixed by the subgroup $\mathrm{GSp}_4(\mathfrak{o})$.

It is known (see, e.g., Table 2 of [PS09]) that $\mathcal{B}_{\Lambda,\theta}(\pi)$ exists for all characters Λ of $F^{\times}\backslash T(F)$. For any such character Λ we let $B_{\phi,\Lambda} \in \mathcal{B}_{\Lambda,\theta}(\pi)$ be the element corresponding to ϕ under some choice of isomorphism $\pi \simeq \mathcal{B}_{\Lambda,\theta}(\pi)$.

For $\ell, m \in \mathbb{Z}$, let

$$(21) \qquad h(\ell,m) = \begin{bmatrix} \varpi^{\ell+2m} & & & \\ & \varpi^{\ell+m} & & \\ & & 1 & \\ & & & \varpi^m \end{bmatrix}.$$

Using the Iwasawa decomposition, one can show that

$$(22) \qquad \mathrm{GSp}_4(F) = \bigsqcup_{\substack{\ell,m\in\mathbb{Z}\\m\geq 0}} R(F)h(\ell,m)\mathrm{GSp}_4(\mathfrak{o});$$

cf. (3.4.2) of [Fur93].

2.2.1. *Growth of the local Bessel function.* We keep the setup as above. We refer to the function $B_{\phi,\Lambda}$ as the local (spherical) Bessel function of type (Λ,θ). Note that this function depends on our choice of realization of π in $\mathcal{B}_{\Lambda,\theta}(\pi)$. However, given π, Λ and θ, this function is canonically specified up to multiples. Our next result bounds the growth of this function along the elements $h(\ell,m)$.

Proposition 2.1. *Let $\phi \in V_\pi$ be a spherical vector. For a character Λ of $F^{\times}\backslash T(F)$, we have $B_{\phi,\Lambda}(h(0,c(\Lambda))) \neq 0$. Furthermore, for non-negative integers ℓ, m satisfying $m \geq c(\Lambda)$, we have*

$$\frac{B_{\phi,\Lambda}(h(\ell,m))}{B_{\phi,\Lambda}(h(0,c(\Lambda)))} \ll (m - c(\Lambda) + 1)^3(\ell+1)^3 q^{-2(m-c(\Lambda))-3\ell/2}.$$

The implied constant is absolute.

2.2.2. *Growth of the local Bessel integral.* We keep the setup as above. For brevity, denote

$$\phi^{(\ell,m)} = \pi(h(\ell,m))\phi.$$

We are interested in bounding the quantity $J_{\Lambda,\theta}(\phi^{(\ell,m)})$ for $m \geq c(\Lambda)$. Due to Proposition 2.1 and the discussion in Section 2.1.3 it suffices to consider the case $\ell = 0$, $m = c(\Lambda)$. We prove the following bound.

Proposition 2.2. *Let $\phi \in V_\pi$ be a spherical vector and Λ be a character of $F^{\times}\backslash T(F)$. If the residue field characteristic q is even and $c(\Lambda) = 0$, assume that $F = \mathbb{Q}_2$. We have*

$$J_{\Lambda,\theta}(\phi^{(0,c(\Lambda))}) \ll (c(\Lambda)+1)^6 q^{-4c(\Lambda)}.$$

The implied constant is absolute.

2.3. **Proof of Proposition 2.1.** Fix a character Λ of $F^{\times}\backslash T(F)$ and recall the definitions of the parameters α, β attached to the representation π, and of the local spherical Bessel function $B_{\phi,\Lambda} \in \mathcal{B}_{\Lambda,\theta}(\pi)$ of type (Λ,θ). It is known that $B_{\phi,\Lambda}(h(\ell,m)) = 0$ whenever $\ell < 0$ or $m < c(\Lambda)$, cf. [Fur93, Lemma 3.4.4] and [Sug85, Lemma 2.5].

2.3.1. *A formula due to Sugano.* In [Sug85, Theorem 2.1 and Proposition 2.5] Sugano proved that $B_{\phi,\Lambda}(h(0,c(\Lambda))) \neq 0$ and provided a formula for the generating function of $B_{\phi,\Lambda}(h(\ell,m))$. We write it down in a form which will be convenient for the proof of Proposition 2.1. Namely:

$$(23) \qquad \sum_{\ell,m\geq 0} B_{\phi,\Lambda}(h(\ell,m+c(\Lambda)))x^m y^\ell = B_{\phi,\Lambda}(h(0,c(\Lambda)))\frac{H(x,y)}{P(x)Q(y)},$$

where

$$P(x) = (1 - \alpha\beta q^{-2}x)(1 - \alpha\beta^{-1}q^{-2}x)(1 - \alpha^{-1}\beta q^{-2}x)(1 - \alpha^{-1}\beta^{-1}q^{-2}x)$$
$$Q(y) = (1 - \alpha q^{-3/2}y)(1 - \beta^{-1}q^{-3/2}y)(1 - \alpha^{-1}q^{-3/2}y)(1 - \beta^{-1}q^{-3/2}y)$$

and

- in case $c(\Lambda) > 0$:

$$H(x,y) = 1 + xq^{-2} - xyq^{-7/2}\sigma(\alpha,\beta) + xy^2q^{-5} + x^2y^2q^{-7}$$

- in case $c(\Lambda) = 0$:

$$
\begin{aligned}
H(x,y) =&\, 1 + xq^{-2}(1 + \delta(\alpha,\beta)) + x^2q^{-4}\left(q^{-1}\left(\frac{K}{\mathfrak{p}}\right) + \delta(\alpha,\beta) + q^{-1/2}\epsilon\sigma(\alpha,\beta)\right) \\
&+ x^3q^{-7}\left(\frac{K}{\mathfrak{p}}\right) \\
&+ y\left[-q^{-2}\epsilon + xq^{-7/2}\left(q^{-1/2}\epsilon\tau(\alpha,\beta) - q^{-1/2}\epsilon - \sigma(\alpha,\beta)\right)\right. \\
&\qquad + x^2q^{-6}\left(\epsilon(\tau(\alpha,\beta) - 1 - \sigma(\alpha,\beta)^2) - q^{1/2}\delta(\alpha,\beta)\sigma(\alpha,\beta)\right) \\
&\qquad \left.- x^3q^{-8}\left(q^{-1/2}\left(\frac{K}{\mathfrak{p}}\right)\sigma(\alpha,\beta) + \epsilon\right)\right] \\
&+ y^2\left[-q^{-4}\left(\frac{K}{\mathfrak{p}}\right) + xq^{-5}\left(1 + q^{-1}\left(\frac{K}{\mathfrak{p}}\right)(\tau(\alpha,\beta) - 2)\right)\right. \\
&\qquad + x^2q^{-7}\left(1 + \delta(\alpha,\beta) + q^{-1}\left(\frac{K}{\mathfrak{p}}\right)(\sigma(\alpha,\beta)^2 - 2\tau(\alpha,\beta) + 2)\right) \\
&\qquad \left.+ x^3q^{-9}\left(q^{-1}\left(\frac{K}{\mathfrak{p}}\right) - q^{-1}\left(\frac{K}{\mathfrak{p}}\right)\delta(\alpha,\beta) + q^{-1}\epsilon^2\right)\right]
\end{aligned}
$$

where

$$\delta(\alpha,\beta) = \frac{q^{1/2}}{q + \left(\frac{K}{\mathfrak{p}}\right)}\left(q^{-1/2}\epsilon^2 + 2q^{-1/2}\left(\frac{K}{\mathfrak{p}}\right) - \epsilon\sigma(\alpha,\beta) - q^{-1/2}\left(\frac{K}{\mathfrak{p}}\right)\tau(\alpha,\beta)\right),$$

$$\epsilon = \begin{cases} 0, & \left(\frac{K}{\mathfrak{p}}\right) = -1, \\ \Lambda(\varpi_K), & \left(\frac{K}{\mathfrak{p}}\right) = 0, \\ \Lambda((\varpi,1)) + \Lambda((1,\varpi)), & \left(\frac{K}{\mathfrak{p}}\right) = 1, \end{cases}$$

with ϖ_K a uniformizer of $\mathfrak{p}_K$, and

$$\sigma(\alpha,\beta) = \alpha + \beta + \alpha^{-1} + \beta^{-1},$$
$$\tau(\alpha,\beta) = \alpha\beta + \alpha\beta^{-1} + \alpha^{-1}\beta + \alpha^{-1}\beta^{-1} + 2.$$

2.3.2. *The required bound.* The proof of Proposition 2.1 will follow from the formula (23). For the sake of brevity, denote:

$$a(m,\ell) := B_{\phi,\Lambda}(h(\ell, m + c(\Lambda)))(B_{\phi,\Lambda}(h(0, c(\Lambda))))^{-1}, \qquad H(x,y) = \sum_{m=0}^{3}\sum_{\ell=0}^{2} h_{m,\ell}x^m y^\ell.$$

Recall that $|\alpha| = |\beta| = 1$. Hence, since $|\epsilon| \le 2$, it follows that $|\delta(\alpha,\beta)| < 28q^{-1/2}$ and we may derive good bounds for $|h_{m,\ell}|$; they are listed in Table 1.

TABLE 1. The values of $\Xi_{m,\ell}$ for which $|h_{m,\ell}| \leq \Xi_{m,\ell}$.

m \ ℓ	0	1	2
0	1	$2q^{-2}$	q^{-4}
1	$q^{-2} + 28q^{-5/2}$	$10q^{-4} + 4q^{-7/2}$	$q^{-5} + 4q^{-6}$
2	$q^{-5} + 36q^{-9/2}$	$130q^{-6}$	$q^{-7} + 28q^{-15/2} + 4q^{-8}$
3	q^{-7}	$2q^{-8} + 4q^{-17/2}$	$5q^{-10} + 28q^{-21/2}$

(A) In case $c(\Lambda) = 0$.

m \ ℓ	0	1	2
0	1	0	0
1	q^{-2}	$4q^{-7/2}$	q^{-5}
2	0	0	q^{-7}
3	0	0	0

(B) In case $c(\Lambda) > 0$.

Note in particular that for all $m, \ell \geq 0$:

$$h_{m,\ell} \ll q^{-2m-\frac{3}{2}\ell}.$$

Using the geometric series expansion of $(P(x)Q(y))^{-1}$ in the formula (23), we obtain

$$a(m,\ell) = \sum_{\tilde{m}=0}^{3} \sum_{\tilde{l}=0}^{2} \left(\sum_{\substack{(m_1,m_2,m_3,m_4) \in \mathbb{Z}_{\geq 0}^4 \\ m_1+m_2+m_3+m_4=\tilde{m}-\tilde{m}}} \alpha^{m_1+m_2-m_3-m_4} \beta^{m_1-m_2+m_3-m_4} \right) \left(\sum_{\substack{(l_1,l_2,l_3,l_4) \in \mathbb{Z}_{\geq 0}^4 \\ l_1+l_2+l_3+l_4=\ell-\tilde{l}}} \alpha^{l_1-l_3} \beta^{l_2-l_4} \right)$$

$$\times h_{\tilde{m},\tilde{l}} q^{-2(m-\tilde{m})} q^{-\frac{3}{2}(\ell-\tilde{l})}$$

Hence, because $|\alpha| = |\beta| = 1$,

(24)
$$|a(m,\ell)q^{2m+\frac{3}{2}\ell}| \leq \sum_{\tilde{m}=0}^{3} \sum_{\tilde{l}=0}^{2} (m-\tilde{m}+1)^3 (\ell-\tilde{l}+1)^3 |h_{\tilde{m},\tilde{l}}| q^{2\tilde{m}+\frac{3}{2}\tilde{l}},$$

and from Table 1 we see that $|h_{\tilde{m},\tilde{l}}| q^{2\tilde{m}+\frac{3}{2}\tilde{l}}$ is bounded above by an absolute and effective constant. This proves that

$$B_{\phi,\Lambda}(h(\ell, m + c(\Lambda))) \ll (m+1)^3 (\ell+1)^3 q^{-2m-\frac{3}{2}\ell} |B_{\phi,\Lambda}(h(0, c(\Lambda)))|$$

where the implied constant is absolute and effective.

2.4. Bounding the local Bessel integral.

2.4.1. *A set of coset representatives.* As a starting point, we write down for each $m \geq 1$ a set of coset representatives for $F^\times U_T(m) \backslash T(F) \simeq F^\times (1 + \mathfrak{p}^m \mathfrak{o}_K) \backslash K^\times$. Recall the definition of Δ_0 from Section 2.1.4.

Lemma 2.3. *For each integer $m > 0$, let the sets D and S_m be as defined below.*

(1) if K/F is an unramified field extension, let $D = \{1\}$ and
$$S_m = \{1 + y\Delta_0 : y \in \mathfrak{o}/\mathfrak{p}^m\} \cup \{x + \Delta_0 : x \in \mathfrak{p}/\mathfrak{p}^m\}.$$

(2) if K/F is a ramified field extension, choose $P_0 = x_0 + y_0\Delta_0 = \left[\begin{smallmatrix} x_0 & y_0 c \\ -y_0 a & x_0 - y_0 b \end{smallmatrix}\right]$ such that $x_0, y_0 \in \mathfrak{o}$ and $\det(P_0) \in \varpi\mathfrak{o}^\times$ (such a choice is possible by (20)). Let $D = \{1, P_0\}$ and
$$S_m = \{1 + y\Delta_0 : y \in \mathfrak{o}/\mathfrak{p}^m\}.$$

(3) if $K = F \times F$, recall that $\sqrt{d} \in \mathfrak{o}^\times$, $\frac{b+\sqrt{d}}{2} \in \mathfrak{o}$, and let $P_0 = 1 + \frac{b+\sqrt{d}}{2\sqrt{d}}(\varpi - 1) + \frac{\varpi - 1}{\sqrt{d}}\Delta_0$. Let $D = \{P_0^n : n \in \mathbb{Z}\}$ and
$$S_m = \{1 + \frac{b + \sqrt{d}}{2}y + y\Delta_0 : y \in \mathfrak{o}/\mathfrak{p}^m, y \notin \frac{-1}{\sqrt{d}} + \mathfrak{p}\}.$$

Then in each case the set $DS_m := \{s_1 s_2 : s_1 \in D, \ s_2 \in S_m\}$ gives a complete set of representatives for the quotient $F^\times U_T(m)\backslash T(F)$.

Proof. It suffices to show that the image of D under (10) or (11) gives a complete set of representatives for $F^\times \mathfrak{o}_K^\times \backslash K^\times$ and the image of S_m under (10) or (11) gives a complete set of representatives for $\mathfrak{o}^\times(1 + \mathfrak{p}^m \mathfrak{o}_K)\backslash \mathfrak{o}_K^\times$.

Inert case: Assume K/F is an unramified extension. Then $F^\times \mathfrak{o}_K^\times = K^\times$, which agrees with $D = \{1\}$. To show that the image of S_m under (10) gives a complete set of representatives for $\mathfrak{o}^\times(1 + \mathfrak{p}^m \mathfrak{o}_K)\backslash \mathfrak{o}_K^\times$, note first that S_m has the correct cardinality by Lemma 3.5.3 of [Fur93] and each element of S_m maps to $\mathfrak{o}_K^\times$. Using (20), it is easy to see any element of $\mathfrak{o}_K^\times$ can upon multiplying by a suitable element of $\mathfrak{o}^\times(1 + \mathfrak{p}^m \mathfrak{o}_K)$ be brought to one of the elements in the image of S_m. The proof is complete.

Ramified case: Assume K/F is a ramified extension. Note that the image of P_0 under (10) is an uniformizer ϖ_K of $\mathfrak{o}_K$. So D maps onto $\{1, \varpi_K\} \simeq F^\times \mathfrak{o}_K^\times \backslash K^\times$. The proof that the image of S_m under (10) gives a complete set of representatives for $\mathfrak{o}^\times(1 + \mathfrak{p}^m \mathfrak{o}_K)\backslash \mathfrak{o}_K^\times$ is essentially identical to the inert case.

Split case: Assume $K = F \times F$. A computation show that the image of P_0 under (11) is the element $(\varpi, 1)$. Therefore D maps onto the set $\{(\varpi^n, 1) : n \in \mathbb{Z}\}$ which is clearly a complete set of representatives for $F^\times(\mathfrak{o}^\times \times \mathfrak{o}^\times)\backslash(F^\times \times F^\times)$. Similarly, the image of S_m under (11) is the set $\{(1 + \sqrt{d}y, 1) : y \in \mathfrak{o}/\mathfrak{p}^m, y \notin \frac{-1}{\sqrt{d}} + \mathfrak{p}\}$ which clearly gives a complete set of representatives for $\mathfrak{o}^\times((1 + \mathfrak{p}^m\mathfrak{o}) \times (1 + \mathfrak{p}^m\mathfrak{o}))\backslash(\mathfrak{o}^\times \times \mathfrak{o}^\times)$. $\qquad\square$

2.4.2. *Some volume computations.* We now calculate some volumes that will appear in our calculation of the local integral.

Lemma 2.4. *Let X, S be two 2×2 matrices. Then*
$$\det(X + S) = \det(X) + \operatorname{tr}({}^t A_S X) + \det(S),$$
where A_S is the adjugate matrix of S. In particular, if S is invertible, then
$$\det(X + S) = \det(X) + (\det S)\operatorname{tr}(S^{-1}X) + \det(S).$$

Proof. Say $X = \left[\begin{smallmatrix} x & y \\ w & z \end{smallmatrix}\right]$ and $S = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. Then
$$\det(X + S) = (x + a)(z + d) - (y + b)(w + c)$$
$$= (ad - bc) + (az + xd - yc - wb) + (xz - yw).$$

Now
$$az + xd - yc - wb = \operatorname{tr}\left[\begin{smallmatrix} xd - wb & * \\ * & az - yc \end{smallmatrix}\right]$$
$$= \operatorname{tr}\left(\left[\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right]\left[\begin{smallmatrix} x & y \\ w & z \end{smallmatrix}\right]\right),$$

which proves the first claim. The second claim comes from the identity
$$^tA_S = (\det S)S^{-1}.$$

$\square$

Lemma 2.5. *Let a, b, c, d be integers. Let*
$$\mathcal{S}_{a,b,c,d} = \{[\begin{smallmatrix} x & y \\ y & z \end{smallmatrix}] \in M_2^{sym}(F) : v(x) \geq a, v(y) \geq b, v(z) \geq c, v(xz - y^2) \geq d\}.$$

Then we have
$$\mathrm{vol}(\mathcal{S}_{a,b,c,d}) \leq q^{-\max\{d/2,b\}-\max\{d,a+c\}} + 2\max\{0, d - a - c\}q^{-d-b}.$$

Proof. We have $\mathcal{S}_{a,b,c,d} \subset S_1 \cup S_2$ where
$$S_1 = \{[\begin{smallmatrix} x & y \\ y & z \end{smallmatrix}] \in M_2^{\mathrm{sym}}(F) : v(x) \geq a, v(y) \geq b, v(z) \geq c, \min\{v(xz), 2v(y)\} \geq d\}$$
and
$$S_2 = \{[\begin{smallmatrix} x & y \\ y & z \end{smallmatrix}] \in M_2^{\mathrm{sym}}(F) : v(x) \geq a, v(y) \geq b, v(z) \geq c, v(xz) < d, xz \in y^2 + \mathfrak{p}^d\}.$$

Volume of S_1: Let $j = \max\{b, \lceil \frac{d}{2} \rceil\}$, so that if $[\begin{smallmatrix} x & y \\ y & z \end{smallmatrix}] \in S_1$ then $y \in \mathfrak{p}^j$. If $a + c \geq d$ then we have
$$\mathrm{vol}(S_1) = \mathrm{vol}(\mathfrak{p}^a \times \mathfrak{p}^j \times \mathfrak{p}^c) \leq q^{-\max\{d/2,b\}-(a+c)}.$$
On the other hand, if $a + c < d$ then let $i = v(x)$. Since $v(xz) \geq d$, we must have $v(z) \geq d - i$, which becomes automatic from the condition $v(z) \geq c$ as soon as $i \geq d - c$. Thus we have
$$\mathrm{vol}(S_1) = \mathrm{vol}(\mathfrak{p}^j)\left(\sum_{i=a}^{d-c-1} \mathrm{vol}(\varpi^i \mathfrak{o}^\times \times \mathfrak{p}^{d-i}) + \mathrm{vol}(\mathfrak{p}^{d-c} \times \mathfrak{p}^c) \right)$$
$$= q^j((d - a - c)(1 - q^{-1})q^{-d} + q^{-d}) \leq (1 + d - a - c)q^{-\max\{d/2,b\}-d}.$$
Combining the cases $a + c \geq d$ and $a + c < d$ together, we find that
$$\mathrm{vol}(S_1) \leq (1 + \max\{0, d - a - c\})q^{-\max\{d/2,b\}-\max\{d,a+c\}}.$$

Volume of S_2: Assume $[\begin{smallmatrix} x & y \\ y & z \end{smallmatrix}] \in S_2$ and let $i = v(x)$ and $j = v(y)$. Since $v(xz) < d$, the condition $xz \in y^2 + \mathfrak{p}^d$ forces $2j = i + v(z) < d$ and in particular $2j - i \geq c$. Moreover, for fixed x, we must have $z \in \frac{y^2}{x} + \mathfrak{p}^{d-i}$. Therefore,
$$\mathrm{vol}(S_2) = \sum_{j \geq b} \mathrm{vol}(\varpi^j \mathfrak{o}^\times) \sum_{i \geq a} \mathrm{vol}(\varpi^i \mathfrak{o}^\times \times \mathfrak{p}^{d-i})\mathbb{1}_{\substack{2j-i \geq c \\ 2j < d}}$$
$$= \sum_{j=b}^{\lfloor \frac{d-1}{2} \rfloor} (1 - q^{-1})q^{-j} \sum_{i=a}^{2j-c} (1 - q^{-1})q^{-d}$$
$$\leq \max\{0, d - c - a\}q^{-b-d}.$$

Therefore, in total we have
$$\mathrm{vol}(\mathcal{S}_{a,b,c,d}) \leq q^{-\max\{d/2,b\}-\max\{d,a+c\}} + \max\{0, d - c - a\}(q^{-\max\{d/2,b\}-d} + q^{-b-d})$$
$$\leq q^{-\max\{d/2,b\}-\max\{d,a+c\}} + 2\max\{0, d - c - a\}q^{-b-d}.$$

$\square$

Definition 2.6. Fix j a non-negative integer. For $Y = [\begin{smallmatrix} x & y \\ y & z \end{smallmatrix}] \in M_2^{\mathrm{sym}}(F)$, define
$$M_1(Y; j) = \min\{0, v(x) + j, v(y), v(z)\}$$
and
$$M_2(Y; j) = \min\{0, j + v(xz - y^2), j + v(x), j + v(y), v(z)\}.$$

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For future use, we note the following inequality, holding for all $Y \in M_2^{\mathrm{sym}}(F)$

(25)
$$2M_1(Y;j) \le M_2(Y,j) \le M_1(Y;j) + j.$$

The second inequality comes from

$$M_2(Y,j) \le \min\{0, j + v(x), j + v(y), v(z)\} \le \min\{0, j + v(x), v(y), v(z)\} + j.$$

To establish the first inequality, observe that

$$v(xz - y^2) \ge \min\{v(x) + v(z), 2v(y)\}$$

and thus

$$M_2(Y,j) \ge \min\{0, v(x) + v(z) + j, 2v(y), j + v(x), j + v(y), v(z)\} \ge 2M_1(Y;j).$$

Lemma 2.7. *For any integers $m_1, m_2 \le 0$ with $2m_1 \le m_2$ define*

$$N(m_1, m_2; j) = \{Y \in M_2^{sym}(F) : M_1(Y;j) = m_1, M_2(Y;j) = m_2\}.$$

Then we have

$$\mathrm{vol}\left(N(m_1, m_2; j)\right) \le (1 + 2\min\{m_2 - 2m_1, -m_2\})q^{-m_1 - m_2 + j}.$$

Proof. If $Y \in N(m_1, m_2; j)$ then by definition of $M_1(Y;j)$ we must have $v(x) \ge m_1 - j$, and by definition of $M_2(Y;j)$ we must also have $v(x) \ge m_2 - j$. Thus $v(x) \ge \max\{m_1, m_2\} - j$. By the same reasoning, it follows that

$$Y \in \mathcal{S}_{a,b,c,d},$$

where

$$a = \max\{m_1, m_2\} - j, b = \max\{m_1, m_2 - j\}, c = \max\{m_1, m_2\}, d = m_2 - j.$$

We have

$$\max\{d/2, b\} = \max\left\{m_1, m_2 - j, \frac{m_2 - j}{2}\right\} = \max\left\{m_1, \frac{m_2 - j}{2}\right\} \ge m_1.$$

Moreover, using the fact that $m_2 \le 0$, we have by (25)

$$\max\{d, a + c\} = \max\{m_2 - j, 2\max\{m_1, m_2\} - j\}$$
$$= \max\{m_2, 2m_1\} - j = m_2 - j.$$

Thus, by Lemma 2.5, we obtain

$$\mathrm{vol}\left(N(m_1, m_2; j)\right) \le (1 + 2\min\{m_2 - 2m_1, -m_2\})q^{-m_1 - m_2 + j}.$$

$\square$

2.4.3. *Proof of Proposition 2.2.* Recall that π is an irreducible, tempered, spherical representation, $\phi \in V_\pi$ is a spherical vector, and Λ is a character of $K^\times$ satisfying $\Lambda|_{F^\times} = 1$.

Our aim is to prove Proposition 2.2. We assume that $c(\Lambda) \ge 1$ because the case $c(\Lambda) = 0$ is known by Theorem 2.1 of [DPSS20]. Our bound will follow from the calculation of explicit coset representatives in Lemma 2.3 together with a bound for the following stable integral. Given a positive integer m and given $t \in T(F)$, we define

$$J_0(\phi^{(0,m)}, t) = \int_{N(F)}^{\mathrm{st}} \Phi_{\phi^{(0,m)}}(nt)\theta^{-1}(n)\,dn.$$

Proposition 2.8. *Let $t \in T(F)$ and let m be a positive integer. Write the Cartan decomposition*

$$(26) \qquad \begin{bmatrix} \varpi^{-2m} & \\ & \varpi^{-m} \end{bmatrix} t \begin{bmatrix} \varpi^{2m} & \\ & \varpi^{m} \end{bmatrix} = UDV$$

with $U, V \in \mathrm{GL}_2(\mathfrak{o})$, $D = \begin{bmatrix} \varpi^i & \\ & \varpi^{i+j} \end{bmatrix}$ and $j \geq 0$. Then we have

$$J_0(\phi^{(0,m)}, t) \ll (m + v(2) + j)^5 q^{-3m - \frac{j}{2}}.$$

Proof. Since $\Phi_{\phi^{(0,m)}}$ is bi-$F^\times h(0,m)\mathrm{GSp}_4(\mathfrak{o})h(0,m)^{-1}$-invariant, we have

$$(27) \qquad J_0(\phi^{(0,m)}, t) = \lim_{k \to \infty} \sum_{\ell', m'} \Phi_{\phi^{(0,m)}}(h(\ell', m')) \int_{N(\mathfrak{p}^{-k}) \cap N(\ell', m'; m, t)} \theta^{-1}(n) \, dn,$$

where

$$N(\ell', m'; m, t) = \{n \in N(F) : h(0,m)^{-1}nth(0,m) \in Z(F)\mathrm{GSp}_4(\mathfrak{o})h(\ell', m')\mathrm{GSp}_4(\mathfrak{o})\}.$$

Now let $n = \begin{bmatrix} 1 & X \\ & 1 \end{bmatrix} \in N(F)$. For convenience, define

$$h(m) = \begin{bmatrix} \varpi^{2m} & \\ & \varpi^{m} \end{bmatrix},$$

so that

$$h(0,m) = \begin{bmatrix} h(m) & \\ & \varpi^{2m}h(m)^{-1} \end{bmatrix}.$$

Then

$$h(0,m)^{-1}nth(0,m)$$
$$= h(0,m)^{-1} \begin{bmatrix} 1 & X \\ & 1 \end{bmatrix} \begin{bmatrix} t & \\ & (\det t)^t t^{-1} \end{bmatrix} h(0,m)$$
$$= \begin{bmatrix} h(m)^{-1} & \\ & \varpi^{-2m}h(m) \end{bmatrix} \begin{bmatrix} 1 & X \\ & 1 \end{bmatrix} \begin{bmatrix} h(m)UDVh(m)^{-1} & \\ & (\det t)h(m)^{-1\,t}U^{-1}D^{-1\,t}V^{-1}h(m) \end{bmatrix} \begin{bmatrix} h(m) & \\ & \varpi^{2m}h(m)^{-1} \end{bmatrix}$$
$$= \begin{bmatrix} U & \\ & {}^t U^{-1} \end{bmatrix} \begin{bmatrix} D & (\det t)YD^{-1} \\ & (\det t)D^{-1} \end{bmatrix} \begin{bmatrix} V & \\ & {}^t V^{-1} \end{bmatrix},$$

where we have set $U^{-1} \begin{bmatrix} \varpi^{-2m} & \\ & \varpi^{-m} \end{bmatrix} X \begin{bmatrix} 1 & \\ & \varpi^m \end{bmatrix}{}^t U^{-1} = Y = \begin{bmatrix} \tilde{x} & \tilde{y} \\ \tilde{y} & \tilde{z} \end{bmatrix}$. Note that there is a unit u such that $\det t = \varpi^{2i+j}u$, and

$$\begin{bmatrix} D & (\det t)YD^{-1} \\ & (\det t)D^{-1} \end{bmatrix} = \begin{bmatrix} 1 & Y \\ & 1 \end{bmatrix} \begin{bmatrix} D & \\ & (\det t)D^{-1} \end{bmatrix}$$
$$= \varpi^i \begin{bmatrix} 1 & Y \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & \varpi^j & & \\ & & \varpi^j & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & u & \\ & & & u \end{bmatrix}.$$

Thus, recalling Definition 2.6, by [DPSS20, Lemma 2.7] we have $n \in N(\ell', m'; m, t)$ if and only if

$$\ell' = j + 2M_1(Y; j) - 2M_2(Y; j)$$

and

$$m' = M_2(Y; j) - 2M_1(Y; j).$$

Henceforth, set $m_1 = -m' - \frac{\ell'}{2} + \frac{j}{2}$ and $m_2 = j - m' - \ell'$, so that $n \in N(\ell', m'; m, t)$ if and only if $Y \in N(m_1, m_2; j)$. Observe that

$$\mathrm{tr}(SX) = \mathrm{tr}(S \begin{bmatrix} \varpi^{2m} & \\ & \varpi^{m} \end{bmatrix} UY^t U \begin{bmatrix} 1 & \\ & \varpi^{-m} \end{bmatrix})$$
$$= \mathrm{tr}(S'Y),$$

where

$$S' = {}^t U \begin{bmatrix} 1 & \\ & \varpi^{-m} \end{bmatrix} S \begin{bmatrix} \varpi^{2m} & \\ & \varpi^{m} \end{bmatrix} U = {}^t U \begin{bmatrix} \varpi^m & \\ & 1 \end{bmatrix} S \begin{bmatrix} \varpi^m & \\ & 1 \end{bmatrix} U.$$

Thus, changing variables $X \mapsto Y$, we have

$$(28) \qquad \int_{N(\mathfrak{p}^{-k}) \cap N(\ell', m'; m, t)} \theta_S^{-1}(n) \, dn = q^{-3m} I^{(k)}(m_1, m_2; j),$$

where

$$I^{(k)}(m_1, m_2; j) = \int_{N(m_1, m_2; j) \cap S_U(k, m)} \psi(-\mathrm{tr}(S'Y)) \, dY$$

and

$$S_U(k, m) = U^{-1} \begin{bmatrix} \varpi^{-2m} & \\ & \varpi^{-m} \end{bmatrix} M_2^{\mathrm{sym}}(\mathfrak{p}^{-k}) \begin{bmatrix} 1 & \\ & \varpi^m \end{bmatrix}^t U^{-1}.$$

Since

$$\left| I^{(k)}(m_1, m_2; j) \right| \leq \mathrm{vol}(N(m_1, m_2; j)),$$

by Lemma 2.7 we obtain

$$(29) \qquad \left| \int_{N(\mathfrak{p}^{-k}) \cap N(\ell', m'; m, t)} \theta_S^{-1}(n) \, dn \right| \leq (1 + 2\min\{m', \ell' + m' - j\}) q^{\frac{3}{2}\ell' + 2m' - \frac{j}{2} - 3m}.$$

By Macdonald's formula [DPSS20, Proposition 2.10] we have for all non-negative integers ℓ', m'

$$(30) \qquad \Phi_{\phi^{(0,m)}}(h(\ell', m')) = \Phi_\phi(h(\ell', m')) \ll (m' + \ell')^2 q^{-(4m' + 3\ell')/2}.$$

The bound (30) follows from the fact that the Macdonald's formula is essentially a double geometric sum in the Satake parameters, of length $\ll m' + \ell'$. Hence combining (30) and (29), every term in (27) is $\ll (m' + \ell')^3 q^{-\frac{j}{2} - 3m}$. It remains to show that the series (27) is actually a finite sum.

From now on, assume that $m_1 + j \leq -2m - 6 - j - 2v(2)$. We claim that for k large enough we have $I^{(k)}(m_1, m_2; j) = 0$. This is enough to prove the Proposition, because then only the terms with $0 \geq m_1 > -2m - 6 - 2j - 2v(2)$ will contribute to (27), but in view of inequality (25) these terms must also satisfy $0 \geq m_2 \geq -4m - 12 - 4j - 4v(2)$, and thus there are $\ll (m + v(2) + j)^2$ such terms.

We split the integral $I^{(k)}(m_1, m_2; j)$ over the following two disjoint ranges:

(1) $v(\mathrm{tr}(S'Y)) < -2$,
(2) $v(\mathrm{tr}(S'Y)) \geq -2$.

For $i \in \{1, 2\}$, let I_i be the integral in the corresponding range. Range (1) is trivially stable by the change of variable $Y \mapsto \lambda Y$ for any $\lambda \in \mathfrak{o}^\times$. Thus

$$I_1 = \int \psi(-\lambda \mathrm{tr}(S'Y)) \, dY.$$

Integrating both sides with respect to $\lambda \in 1 + \mathfrak{p}$ gives $I_1 = 0$.

Next, we claim that range (2) is stable by the change of variables $Y \mapsto Y + \varpi^{-1-v(2)} S'^{-1}$. To prove this, we have three conditions to check:

(1) We have

$$v[\mathrm{tr}(S'(Y + \varpi^{-1-v(2)} S'^{-1}))] = v(\mathrm{tr}(S'Y) + 2\varpi^{-1-v(2)}) \geq -2.$$

(2) We need to check that $M_1(Y + \varpi^{-1-v(2)} S'^{-1}; j) = M_1(Y; j) = m_1$. We have $\det(S') = \varpi^{2m} \det(S) = \varpi^{2m} d$ and by Assumption 14,

$$v(\det(S')) \leq 2m + 2.$$

Moreover, since the entries of S are integers, it is clear from the definition that S' also has integer coefficients. Since

$$S'^{-1} = \frac{1}{\det S'} {}^t A_{S'},$$

where $A_{S'}$ is the adjugate matrix of S', it follows that all the entries of $\varpi^{-1-v(2)}S'^{-1}$ have valuation larger than $-2m - 3 - v(2)$. Since we are assuming $M_1(Y; j) = m_1 \leq -2m - 6 - 2j - 2v(2) < -2m - 3 - v(2) - j$, it follows that $M_1(Y + \varpi^{-1-v(2)}S'^{-1}; j) = M_1(Y; j)$, as required.

(3) We need to check that $M_2(Y + \varpi^{-1-v(2)}S'^{-1}; j) = M_2(Y; j) = m_2$. By Lemma 2.4 we have

$$\det(Y + \varpi^{-1-v(2)}S'^{-1}) = \det(Y) + \varpi^{-1-v(2)}\det(S')^{-1}\mathrm{tr}(S'Y) + \varpi^{-2-2v(2)}\det(S')^{-1}.$$

In particular,

$$v(\det(Y + \varpi^{-1-v(2)}S'^{-1})) = v(\det(Y))$$

unless $v(\det(Y)) \geq -2m - 5 - 2v(2)$, in which case it does not contribute to $M_2(Y; j)$ anyway since by (25), we have $M_2(Y; j) = m_2 \leq m_1 + j < -2m - 5 - j - 2v(2)$. In both cases, we have $M_2(Y + \varpi^{-1-v(2)}S'^{-1}; j) = M_2(Y; j)$.

Thus, we obtain

$$I_2 = \psi(2\varpi^{-1-v(2)})I_2.$$

Since $2\varpi^{-1-v(2)}$ generates $\mathfrak{p}^{-1}/\mathfrak{o}$ and since ψ is trivial on $\mathfrak{o}$ but not trivial on $\mathfrak{p}^{-1}$, we have $\psi(2\varpi^{-1-v(2)}) \neq 1$. This implies $I_2 = 0$. $\qquad\square$

We now proceed to prove Proposition 2.2 in all cases – split, inert and ramified. Until the end of the section, we set $m = c(\Lambda)$. We remind the reader that Assumptions 14 are in force throughout. By $U_T(m)$-invariance, we have

$$(31) \qquad J_{\Lambda,\theta}(\phi^{(0,m)}) = \mathrm{vol}(\mathfrak{o}^\times \backslash U_T(m)) \sum_i \Lambda^{-1}(t_i) J_0(\phi^{(0,m)}, t_i),$$

where the i-sum ranges over representatives t_i of $F^\times U_T(m) \backslash T(F)$. Note that $\mathrm{vol}(\mathfrak{o}^\times \backslash U_T(m)) \asymp q^{-m}$.

Proof in the inert case: Assume K/F is an unramified field extension. Let t be a representative as in Lemma 2.3. Consider first the case

$$t = \begin{bmatrix} 1 & yc \\ -ya & 1-yb \end{bmatrix}.$$

Write

$$\begin{bmatrix} \varpi^{-2m} & \\ & \varpi^{-m} \end{bmatrix} t \begin{bmatrix} \varpi^{2m} & \\ & \varpi^m \end{bmatrix} = \begin{bmatrix} 1 & \varpi^{-m}yc \\ -\varpi^m ya & 1-yb \end{bmatrix} = UDV$$

as in (26). Noting that $v(y) - m \leq 0$, we have

$$i = \min\{v(1), v(y) - m, v(ya) + m, v(1 - yb)\}$$
$$= v(y) - m.$$

On the other hand, since $t \in S_m$ and the image of S_m under (10) is contained in $\mathfrak{o}_K^\times$, we have

$$2i + j = v(\det(t)) = v(\mathrm{N}(1 + y\delta_0)) = 0.$$

Therefore

$$j = 2m - 2v(y).$$

Thus by Proposition 2.8 we have $J_0(\phi^{(0,m)}, t) \ll (m + v(2))^5 q^{-4m+v(y)}$. Since there are $\asymp q^{m-v(y)}$ representatives of this form, their total contribution (taking into account the volume) to (31) is $\ll (m + v(2))^5 q^{-4m}$. Summing this over $0 \leq v(y) \leq m$ gives $\ll m(m + v(2))^5 q^{-4m}$. Next, consider the case

$$t = \begin{bmatrix} x & c \\ -a & x-b \end{bmatrix}$$

with $x \in \mathfrak{p}$. Then similarly as before, we find $i = -m$ and $2i + j = 0$. Thus by Proposition 2.8 we have $J_0(\phi^{(0,m)}, t) \ll (m + v(2))^5 q^{-4m}$. Since there are $\asymp q^{m-1}$ representatives of this form, their total contribution to (31) is $\ll (m + v(2))^5 q^{-4m-1}$.

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Adding up the contributions of all the representatives thus gives

$$J_{\Lambda,\theta}(\phi^{(0,m)}) \ll m(m + v(2))^5 q^{-4m}.$$

<u>Proof in the ramified case:</u> Now assume K/F is a ramified field extension. Let t be a representative as in Lemma 2.3. Consider first the case

$$t = \begin{bmatrix} 1 & yc \\ -ya & 1-yb \end{bmatrix} \in S_m.$$

Then the exact same argument as in the inert case goes through. Next, consider the case

$$t = (x_0 + y_0\Delta_0)(1 + y\Delta_0) = \begin{bmatrix} x_0 - yy_0ac & c[y_0 + y(x_0 - y_0b)] \\ -a[y_0 + y(x_0 - y_0b)] & x_0 + y[y_0(b^2 - ac) - bx_0] - by_0 \end{bmatrix}.$$

Recall that x_0, y_0 are any two elements of $\mathfrak{o}$ such that

$$(32) \qquad v(x_0(x_0 - y_0b) + y_0^2ac) = 1.$$

We shall now choose some convenient x_0, y_0. Let us now distinguish some cases. Consider first the case $v(a) = 1$. Then we can take $y_0 = 1$ and $x_0 = y_0b$. Thus $v([y_0 + y(x_0 - y_0b)]) = 0$, and hence $i = -m$. Furthermore

$$2i + j = v(\det(t)) = v(\det(P_0)) = 1,$$

and hence $j = 2m + 1$. From there on, the argument proceeds as in the second case of the inert case, giving a final contribution $\ll (m + v(2))^5 q^{-4m-\frac{1}{2}}$

Next suppose $v(a) > 1$. First, we claim that $v(b) = 0$. Indeed, if $b = \varpi b'$ with $b' \in \mathfrak{o}$ then we have $d = \varpi^2 d'$ where $d' = b'^2 - 4\frac{a}{\varpi^2}c \in \mathfrak{o}$ and thus d does not generate the discriminant of K/F, contradicting Assumption 14. Thus we can take $y_0 = 1$ and $x_0 = \varpi + b$. Then $v(y_0 + y(x_0 - y_0b)) = 0$ again and the rest of the argument is the same as the case $v(a) = 1$.

Finally consider the case $v(a) = 0$. For (32) to hold, we must have $v(y_0) = v(x_0 - y_0b) = 0$. Since $y \in \mathfrak{o}/\mathfrak{p}^m$, for each $0 \le k \le m$ there are exactly $q^k(1 - q^{-1})$ choices of y such that $v([y_0 + y(x_0 - y_0b)]) = k$. From there on, the argument proceeds as in the first case of the inert case, giving a final contribution $\ll m(m + v(2))^5 q^{-4m-\frac{1}{2}}$.

Adding up the contributions of all the representatives again gives

$$J_{\Lambda,\theta}(\phi^{(0,m)}) \ll m(m + v(2))^5 q^{-4m}.$$

<u>Proof in the split case:</u> Finally assume $K = F \times F$. We start with a technical lemma.

Lemma 2.9. *Assume $v(2) > 0$. Then we have $v(\sqrt{d} - b), v(\sqrt{d} + b) \ge v(2)$.*

Proof. We have $d \in b^2 + 4\mathfrak{o}$ thus

$$(33) \qquad v(\sqrt{d} + b) + v(\sqrt{d} - b) \ge v(4).$$

But

$$(34) \qquad v(\sqrt{d} + b) = v(\sqrt{d} - b + 2b) \ge \min\{v(2), v(\sqrt{d} - b)\}.$$

If $v(\sqrt{d} - b) \le v(2)$ then equation (33) implies $v(\sqrt{d} + b) \ge v(2)$, while if $v(\sqrt{d} - b) \ge v(2)$ then equation (34) implies $v(\sqrt{d} + b) \ge v(2)$. The same reasoning gives $v(\sqrt{d} - b) \ge v(2)$. $\qquad\square$

Now let t be a representative as in Lemma 2.3, and write $t = x + y\Delta = \begin{bmatrix} x+y\frac{b}{2} & yc \\ -ya & x-y\frac{b}{2} \end{bmatrix}$ for some $x, y \in F$. Since by (11) the image of P_0 is $(\varpi^n, 1)$ and the image of S_m is $(\mathfrak{o}/\mathfrak{p}^m)^\times \times \{1\}$, we have

$$\begin{cases} x + y\frac{\sqrt{d}}{2} = \varpi^k u, \\ x - y\frac{\sqrt{d}}{2} = 1 \end{cases}$$

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for some $k \in \mathbb{Z}$ and $u \in (\mathfrak{o}/\mathfrak{p}^m)^\times$. Equivalently,

$$\begin{cases} x = \frac{\varpi^k u + 1}{2}, \\ y = \frac{\varpi^k u - 1}{\sqrt{d}}. \end{cases}$$

We start with the case $k < 0$. Then $v(y) = k$. Furthermore, we claim that $v(x \pm y\frac{b}{2}) \geq k$. If $v(2) = 0$ this is obvious. On the other hand, we have

$$\begin{aligned} x + y\frac{b}{2} &= \frac{\varpi^k u + 1}{2} + b\frac{\varpi^k u - 1}{2\sqrt{d}} \\ &= \frac{(\sqrt{d} + b)\varpi^k u + (\sqrt{d} - b)}{2\sqrt{d}}. \end{aligned}$$

If $v(2) > 0$, Lemma 2.9 proves the claim, and similarly for $x - y\frac{b}{2}$. It follows that

$$\begin{aligned} i &= \min\left\{v\left(x + y\frac{b}{2}\right), v(y) - m, v(ya) + m, v\left(x - y\frac{b}{2}\right)\right\} \\ &= k - m. \end{aligned}$$

Moreover, we have

$$2i + j = v\left(x^2 - \frac{d}{4}y^2\right) = v(\varpi^k u) = k.$$

Therefore, $j = 2m - k$ and by Proposition 2.8 we have

$$J_0(\phi^{(0,m)}, t) \ll (m + v(2) - k)^5 q^{-4m + \frac{k}{2}}.$$

Summing over $k < 0$ and $u \in (\mathfrak{o}/\mathfrak{p}^m)^\times$ and multiplying by $\mathrm{vol}(\mathfrak{o}^\times \backslash U_T(m))$ gives a contribution $\ll (m + v(2))^5 q^{-4m - \frac{1}{2}}$.

Consider now the case $k > 0$. Then $v(y) = 0$ and, by the same reasoning as as above, $v(x \pm y\frac{b}{2}) \geq 0$, hence

$$i = -m$$

and

$$2i + j = k.$$

Therefore, $j = 2m + k$, and from then on this case is completely analogous to the case $k < 0$ after changing k to $-k$.

Finally, consider the case $k = 0$. We again have $v(x \pm y\frac{b}{2}) \geq 0$. Since we can always choose the representative $u \in (\mathfrak{o}/\mathfrak{p}^m)^\times$ such that $v(y) \leq m$, it follows that $i = v(y) - m$ and $2i + j = 0$. Therefore by Proposition 2.8 we have

$$J_0(\phi^{(0,m)}, t) \ll (m + v(2))^5 q^{-4m + v(y)}.$$

Since there are $\asymp q^{m - v(y)}$ such representatives, we get as before a contribution $\ll m(m + v(2))^5 q^{-4m}$.

Adding up the contributions of all the representatives thus gives

$$J_{\Lambda, \theta}(\phi^{(0,m)}) \ll m(m + v(2))^5 q^{-4m}.$$

3. Bounds on Fourier coefficients and sup-norms of Siegel cusp forms

In this section we relate the Fourier coefficients of Siegel cusp forms of degree 2 to global Bessel coefficients and the local quantities studied in the previous section. We go on to prove the main theorems stated in the introduction.

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3.1. Siegel cusp forms and representations.

We begin with recalling classical Siegel cusp forms and their adelizations. For brevity, denote $G(R) := \mathrm{GSp}_4(R)$ for each ring R and let $\Gamma = \mathrm{Sp}_4(\mathbb{Z})$. Let k be a positive integer.

Let $S_k(\Gamma)$ be the space of holomorphic Siegel cusp forms of degree 2 and weight k with respect to Γ. Hence, if $F \in S_k(\Gamma)$, then for all $\gamma \in \Gamma$ we have $F|_k\gamma = F$, where

$$(35) \qquad (F|_k g)(Z) := \mu(g)^k j(g, Z)^{-k} F(g\langle Z\rangle)$$

for $g \in G(\mathbb{R})^+$ and $Z \in \mathbb{H}_2$, the Siegel upper half space; moreover F vanishes at the cusps. For a precise formulation of this cusp vanishing condition, see [Fre91, I.4.6]; for definitions and basic properties of Siegel cusp forms we refer the reader to [AZ95].

For $F \in S_k(\Gamma)$, we define the adelization ϕ_F of F to be the function on $G(\mathbb{A})$ defined by

$$(36) \qquad \phi_F(\gamma h_\infty k_0) = \mu(h_\infty)^k j(h_\infty, iI_2))^{-k} F(h_\infty\langle iI_2\rangle))$$

where $\gamma \in G(\mathbb{Q}), h_\infty \in G(\mathbb{R})^+$ and $k_0 \in \prod_{p<\infty} G(\mathbb{Z}_p)$. Then ϕ_F is a well-defined function on the whole of $G(\mathbb{A})$ by strong approximation, and is a cuspidal automorphic form.

It is well-known that ϕ_F generates an irreducible representation if and only if F is an eigenform of all the Hecke operators [NPS13]. We denote this representation by π_F. Now suppose that F is a Hecke eigenform. Since F is of full level, it is either of general type or of Saito–Kurokawa type (see, e.g., Proposition 2.3.1 of [PS22]). The representation π_F is an irreducible cuspidal automorphic representation of $G(\mathbb{A})$ of trivial central character; writing $\pi_F = \otimes_v \pi_v$, we have

- The archimedean component π_∞ is a holomorphic discrete series representation with scalar minimal K-type determined by the weight k.
- If F is of Saito–Kurokawa type, then for a prime number p, the representation π_p is of type IIb according to Table A.1 of [RS07]. Note that these are non-tempered, non-generic representations.
- If F is of general type, then for a prime number p, π_p is a *tempered* representation of Type I in the notation of [RS07].

For $F \in S_k(\Gamma)$ we define the Petersson norm

$$(37) \qquad \|F\|_2^2 = \langle F, F\rangle = \int\limits_{\Gamma\backslash\mathbb{H}_2} |F(Z)|^2 (\det Y)^{k-3}\, dX\, dY.$$

For any $\phi \in L^2(\mathbb{A}^\times G(\mathbb{Q})\backslash G(\mathbb{A}))$, let $\langle \phi, \phi\rangle = \int_{\mathbb{A}^\times G(\mathbb{Q})\backslash G(\mathbb{A})} |\phi(g)|^2\, dg$. We have the convenient relation

$$(38) \qquad \frac{\langle F, F\rangle}{\mathrm{vol}(\mathrm{Sp}(4, \mathbb{Z})\backslash \mathbb{H}_2)} = \frac{\langle \phi_F, \phi_F\rangle}{\mathrm{vol}(Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A}))};$$

c.f. Section 4.1 of [AS01].

3.2. Bessel periods and Fourier coefficients.

Given a fundamental discriminant $d < 0$, define

$$(39) \qquad S_d = \begin{cases} \begin{bmatrix} \frac{-d}{4} & 0 \\ 0 & 1 \end{bmatrix} & \text{if } d \equiv 0 \pmod 4, \\[2ex] \begin{bmatrix} \frac{1-d}{4} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} & \text{if } d \equiv 1 \pmod 4. \end{cases}$$

Given S_d as above, let the group $T_d = T_{S_d}$ over $\mathbb{Q}$ be defined in the same way as in (12). So $T_d \simeq K^\times$ where $K = \mathbb{Q}(\sqrt{d})$. Note that the matrix S_d above satisfies the standard assumptions (14) at every finite prime.

Let $\psi : \mathbb{Q}\backslash\mathbb{A} \to \mathbb{C}^\times$ be the character such that $\psi(x) = e^{2\pi i x}$ if $x \in \mathbb{R}$ and $\psi(x) = 1$ for $x \in \mathbb{Z}_p$. One obtains a character θ_{S_d} of $N(\mathbb{Q})\backslash N(\mathbb{A})$ by $\theta_{S_d}(\begin{bmatrix} 1 & X \\ & 1\end{bmatrix}) = \psi(\mathrm{Tr}(S_d X))$. Let Λ be a character of

$K^\times \backslash \mathbb{A}_K^\times$ such that $\Lambda|_{\mathbb{A}^\times} = 1$. Then for a measurable function $\phi : \mathbb{A}^\times G(\mathbb{Q}) \backslash G(\mathbb{A}) \to \mathbb{C}$, we define the Bessel period

$$
(40) \qquad B(\phi, \Lambda) = \int_{\mathbb{A}^\times T_d(\mathbb{Q}) \backslash T_d(\mathbb{A})} \int_{N(\mathbb{Q}) \backslash N(\mathbb{A})} \phi(tn) \Lambda^{-1}(t) \theta_{S_d}^{-1}(n) \, dn \, dt,
$$

where we give the adelic groups the Tamagawa measure.

In the special case where $\phi = \phi_F$ with F a Siegel cusp form of degree 2, the function $B(\phi_F, \Lambda)$ captures information about all the Fourier coefficients of F. Let us now make this precise. Let $F \in S_k(\Gamma)$. Then F has a Fourier expansion

$$
(41) \qquad F(Z) = \sum_{T \in \Lambda_2} a(F, T) e^{2\pi i \mathrm{Tr}(TZ)}
$$

with

$$
(42) \qquad \Lambda_2 = \left\{ \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} : \quad a, b, c \in \mathbb{Z}, \quad a > 0, \quad d := b^2 - 4ac < 0 \right\}.
$$

For a matrix $T = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} \in \Lambda_2$, we distinguish its content $c(T) = \gcd(a, b, c)$ and discriminant $\mathrm{disc}(T) = -4 \det(T)$. Clearly, $c(T)^2$ divides $\mathrm{disc}(T)$.

From the invariance property of F, we easily see that its Fourier coefficients satisfy the relation

$$
(43) \qquad a(F, T) = a(F, {}^t A T A)
$$

for any $A \in \mathrm{SL}_2(\mathbb{Z})$. Therefore, for $T_1, T_2 \in \Lambda_2$, we consider an equivalence relation

$$
T_1 \sim T_2 \qquad \Longleftrightarrow \qquad \text{there exists } A \in \mathrm{SL}_2(\mathbb{Z}) \text{ such that } {}^t A T_1 A = T_2
$$

and say that the matrices T_1, T_2 satisfying this relation are $\mathrm{SL}_2(\mathbb{Z})$-equivalent. The equivalence class of T will be denoted by $[T]$. Note that:

- the Fourier coefficient $a(F, T)$ depends only on the $\mathrm{SL}_2(\mathbb{Z})$-equivalence class of T;
- if $T_1 \sim T_2$, then $\mathrm{disc}(T_1) = \mathrm{disc}(T_2)$ and $c(T_1) = c(T_2)$.

Given any $T \in \Lambda_2$, it is clear that there exists a fundamental discriminant $d < 0$ and positive integers L, M such that $c(T) = L$, $\mathrm{disc}(T) = dL^2 M^2$; furthermore d, L, M are uniquely determined by T. For a fundamental discriminant $d < 0$ and positive integers L, M, we let $H(dM^2; L)$ denote the set of $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes of matrices in Λ_2 such that $c(T) = L$ and $\mathrm{disc}(T) = dL^2 M^2$. It is clear that given $[X] \in H(dM^2; L)$ and $F \in S_k(\Gamma)$, the notation $a(F, [X])$ is well-defined.

We define

$$
\mathrm{Cl}_d(M) = T_d(\mathbb{A})/T_d(\mathbb{Q}) T_d(\mathbb{R}) \prod_{p < \infty} U_{T_d}(m_p)
$$

where we write $M = \prod_{p < \infty} p^{m_p}$ and the subgroup $U_{T_d}(m_p) \subset T_d(\mathbb{Z}_p)$ is defined in Section 2.1.4. It is well known that $\mathrm{Cl}_d(M)$ can be naturally identified with the class group of the unique order of discriminant $D = dM^2$ in $\mathbb{Q}(\sqrt{d})$; in particular, $\mathrm{Cl}_d(1)$ is canonically isomorphic to the ideal class group of $\mathbb{Q}(\sqrt{d})$. From Proposition 5.3 of [PSS17] we have $|\mathrm{Cl}_d(M)| = \frac{M}{u(d)} |\mathrm{Cl}_d(1)| \prod_{p|M} \left(1 - \left(\frac{d}{p} \right) p^{-1} \right)$ where $u(-3) = 3$, $u(-4) = 2$ and $u(d) = 1$ for other d. In particular this shows that

$$
(44) \qquad (|d|^{1/2} M)^{1-\epsilon} \ll_\epsilon |\mathrm{Cl}_d(M)| \ll_\epsilon (|d|^{1/2} M)^{1+\epsilon}.
$$

Section 5 of [PSS17] constructs for each $c \in T_d(\mathbb{A})$ a matrix $\phi_{L,M}(c) \in \Lambda_2$ such that $\phi_{L,M}(c) = \begin{bmatrix} L & \\ & L \end{bmatrix} \begin{bmatrix} M & \\ & 1 \end{bmatrix} S_c \begin{bmatrix} M & \\ & 1 \end{bmatrix}$ with $\mathrm{disc}(S_c) = d$ and the $(2,2)$-coefficient of S_c is 1 modulo M; this implies that

$$
c(\phi_{L,M}(c)) = L, \quad \mathrm{disc}(\phi_{L,M}(c)) = dL^2 M^2.
$$

The matrix $\phi_{L,M}(c)$ depends on some choices, but its class in $H(dM^2; L)$ is independent of those choices. We recall a result from [PSS17].

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Lemma 3.1 (Prop 5.3 of [PSS17]). *For each pair of positive integers L, M, the map $c \mapsto [\phi_{L,M}(c)]$ gives a bijection from $\mathrm{Cl}_d(M)$ to $H(dM^2; L)$.*

Write $L = \prod_{p<\infty} p^{\ell_p}$, $M = \prod_{p<\infty} p^{m_p}$. Define the element $H(L, M) \in G(\mathbb{A})$ via

$$H(L, M)_p := \begin{cases} h_p(\ell_p, m_p), & p \mid LM \\ 1, & p \nmid LM \text{ or } p = \infty \end{cases},$$

where the element $h_p(\ell_p, m_p)$ was defined in (21); note that we now add the subscript p to avoid confusion. For an automorphic form ϕ on $G(\mathbb{A})$, we let $\phi^{L,M}$ be the automorphic form obtained by right-translation by $H(L, M)$, i.e., $\phi^{L,M}(g) := \phi(gH(L, M))$, for all $g \in G(\mathbb{A})$. We can now state the relation between Bessel coefficients and Fourier coefficients.

Lemma 3.2. *Let $F \in S_k(\Gamma)$ and ϕ_F be its adelization. For any character Λ of $\mathrm{Cl}_d(M)$, we have*

$$B(\phi_F^{L,M}, \Lambda) = \frac{2}{|\mathrm{Cl}_d(M)|}(LM)^{-k}e^{-2\pi \mathrm{Tr}(S_d)} \sum_{c \in \mathrm{Cl}_d(M)} \Lambda^{-1}(c)a(F, \phi_{L,M}(c)).$$

Proof. This is a standard calculation; see, e.g., the proof of [Mar21, Theorem 3]. $\qquad\square$

Applying orthogonality relations and noting the bijection between $\mathrm{Cl}_d(M)$ and $H(dM^2; L)$, we obtain the key formula

$$(45) \qquad \sum_{[T] \in H(dM^2; L)} |a(F, [T])|^2 = \frac{1}{4}(LM)^{2k}e^{4\pi \mathrm{Tr}(S_d)}|\mathrm{Cl}_d(M)| \sum_{\Lambda \in \widehat{\mathrm{Cl}_d(M)}} |B(\phi_F^{L,M}, \Lambda)|^2.$$

3.3. The refined GGP identity and bounds on Fourier coefficients. In the case that $F \in S_k(\Gamma)$ is a Hecke eigenform, we can refine the formula (45). In this case, the adelization of F generates an irreducible automorphic representation π_F. Let $K = \mathbb{Q}(\sqrt{d})$ and Λ be a character of $K^\times \backslash \mathbb{A}_K^\times$ such that $\Lambda|_{\mathbb{A}^\times} = 1$. Liu [Liu16] formulated a precise refinement of the Gan–Gross–Prasad conjecture for $(\mathrm{SO}(5), \mathrm{SO}(2))$ which applies in particular to the Bessel periods $B(\phi, \Lambda)$ where ϕ is any automorphic form in the space of π_F. Recently Furusawa and Morimoto have proved this conjecture [FM22] for tempered representations; in particular their result applies to π_F whenever F is not of Saito–Kurokawa type (the Saito–Kurokawa case is easier and can be dealt with separately; see, e.g., [Qiu21]).

Theorem 3.3 (Furusawa–Morimoto, Theorem 1.2 of [FM22]). *Let $\pi_F = \otimes_v \pi_v$ and Λ be as above and assume that π_F is tempered. Let $\phi = \otimes_v \phi_v$ be an automorphic form in the space of π_F. Let S be a set of places including the archimedean place such that $\pi_v, K_v, \phi_v,$ and Λ_v are all unramified for $v \notin S$. Then*

$$(46) \qquad \frac{|B(\phi, \Lambda)|^2}{\langle \phi, \phi \rangle} = \frac{C_T}{S_{\pi_F}} \frac{\xi(2)\xi(4)L^S(1/2, \pi_F \times \mathcal{AI}(\Lambda^{-1}))}{L^S(1, \pi_F, \mathrm{Ad})L^S(1, \chi_d)} \prod_{v \in S} J_{\Lambda_v, \theta_v}(\phi_v),$$

where $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ denotes the completed Riemann zeta function, C_T is a constant relating our choice of local and global Haar measures, S_{π_F} denotes a certain integral power of 2, related to the Arthur parameter of π_F, and $J_{\Lambda_v, \theta_v}(\phi_v)$ equals the local Bessel integral defined in Section 2.1.3.

Proposition 3.4. *Let $F \in S_k(\Gamma)$ and let ϕ_F be its adelization. Assume that F is not a Saito–Kurokawa lift and let π_F be the automorphic representation generated by F. Let L, M be positive integers and let Λ be a character of $\mathrm{Cl}_d(M)$. Then*

$$\frac{|B(\phi_F^{L,M}, \Lambda)|^2}{\langle \phi_F, \phi_F \rangle} \ll_\epsilon \frac{(LMd)^\epsilon}{M^4 L^3} \frac{(4\pi)^{2k}|d|^{k-2}e^{-4\pi \mathrm{Tr}(S_d)}}{\Gamma(2k-1)} \frac{L(1/2, \pi_F \times \mathcal{AI}(\Lambda^{-1}))}{L(1, \pi_F, \mathrm{Ad})}.$$

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Proof. We return to the setup of Section [3.2] but assume also that F is a Hecke eigenform. Write $\phi_F = \otimes_v \phi_v$, $\pi_F = \otimes_v \pi_v$ and $\phi_F^{L,M} = \otimes_v \phi_v^{(l_v, m_v)}$. By the uniqueness of local and global Bessel functions, we have the following relation (see also Lemma 5 in [Mar21]).

$$(47) \qquad B(\phi_F^{L,M}, \Lambda) = \left(\prod_{p \mid LM} \frac{B_{\phi_p, \Lambda_p}(h_p(\ell_p, m_p))}{B_{\phi_p, \Lambda_p}(h_p(0, c(\Lambda_p)))} \right) B(\phi_F^{1, C(\Lambda)}, \Lambda)$$

where $C(\Lambda) = \prod_{p \mid M} p^{c(\Lambda_p)}$. Using Proposition [2.1], we therefore obtain

$$(48) \qquad |B(\phi_F^{L,M}, \Lambda)|^2 \ll_\epsilon (LM)^\epsilon \frac{C(\Lambda)^4}{M^4 L^3} |B(\phi_F^{1, C(\Lambda)}, \Lambda)|^2.$$

On the other hand, using (46), Proposition [2.2], and the computation $C_T J_\infty \asymp \frac{(4\pi)^{2k} |d|^{k-2} e^{-4\pi \mathrm{Tr}(S_d)}}{\Gamma(2k-1) L(1, \chi_d)}$ which follows from [DPSS20, Sec. 3.5] we have

$$(49) \qquad \frac{|B(\phi_F^{1, C(\Lambda)}, \Lambda)|^2}{\langle \phi_F, \phi_F \rangle} \ll_\epsilon C(\Lambda)^{-4+\epsilon} \frac{(4\pi)^{2k} |d|^{k-2} e^{-4\pi \mathrm{Tr}(S_d)}}{\Gamma(2k-1)} \frac{L(1/2, \pi_F \times \mathcal{AI}(\Lambda^{-1}))}{L(1, \pi_F, \mathrm{Ad}) L(1, \chi_d)^2}$$

The proof follows by combining (48), (49) and the well-known fact $L(1, \chi_d) \gg_\epsilon |d|^{-\epsilon}$. $\qquad \square$

We are ready to prove our main theorem.

Theorem 3.5. *Let $F \in S_k(\Gamma)$ be a Hecke eigenform that is not a Saito–Kurokawa lift and let π_F be the automorphic representation generated by F. Let $d < 0$ be a fundamental discriminant and let L, M be positive integers.*

(1) For each character Λ of $\mathrm{Cl}_d(M)$, we have

$$\left| \sum_{c \in \mathrm{Cl}_d(M)} \Lambda(c) a(F, \phi_{L,M}(c)) \right|^2 \ll_\epsilon \langle F, F \rangle \frac{(4\pi)^{2k} L^{2k-3+\epsilon} |dM^2|^{k-1+\epsilon}}{\Gamma(2k-1)} \frac{L(1/2, \pi_F \times \mathcal{AI}(\Lambda))}{L(1, \pi_F, \mathrm{Ad})}.$$

(2) We have the bound

$$\sum_{[T] \in H(dM^2; L)} |a(F, [T])|^2 \ll_\epsilon \langle F, F \rangle \frac{(4\pi)^{2k}}{\Gamma(2k-1)} L^{2k-3+\epsilon} |dM^2|^{k-\frac{3}{2}+\epsilon} \sum_{\Lambda \in \widehat{\mathrm{Cl}_d(M)}} \frac{L(1/2, \pi_F \times \mathcal{AI}(\Lambda))}{L(1, \pi_F, \mathrm{Ad})}.$$

Proof. The first assertion follows from (38), (44), Lemma [3.2] and Proposition [3.4]. The second assertion follows from (38), (44), (45) and Proposition [3.4]. $\qquad \square$

Remark 3.6. *Note that Theorem [1.1] is just a restatement of part (2) of Theorem [3.5]. In the terminology of Theorem [1.1] we have $D = dM^2$, $H_D = \mathrm{Cl}_d(M)$.*

Corollary 3.7. *Let $F \in S_k(\Gamma)$ be a Hecke eigenform that is not a Saito–Kurokawa lift. Let $d < 0$ be a fundamental discriminant and M a positive integer. Assume that for some real numbers α, β, the bound*

$$(50) \qquad \frac{L(1/2, \pi_F \times \mathcal{AI}(\Lambda))}{L(1, \pi_F, \mathrm{Ad})} \ll_\epsilon k^{\alpha+\epsilon} (|d| M^2)^{\beta+\epsilon}$$

holds for all $\Lambda \in \mathrm{Cl}_d(M)$. Then

$$\frac{1}{\langle F, F \rangle} \sum_{[T] \in H(dM^2; L)} |a(F, [T])|^2 \ll_\epsilon \frac{k^{1/2+\alpha+\epsilon} (2\pi)^{2k}}{\Gamma(k)^2} L^{-1} |dM^2 L^2|^{k-1+\beta+\epsilon}.$$

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Furthermore, for any $T \in \Lambda_2$, we have

$$\frac{|a(F,T)|}{\|F\|_2} \ll_\epsilon \frac{k^{\frac{1}{4}+\frac{\alpha}{2}}(4\pi)^k}{\Gamma(k)} c(T)^{-\frac{1}{2}-\beta} \det(T)^{\frac{k-1+\beta}{2}}.$$

Proof. The first assertion follows by substituting the bound (50) into Theorem 1.1 and using the duplication formula for the Gamma function. For the second assertion, we note that for any $T \in \Lambda_2$ we have $[T] \in H(dM^2; L)$ where we put $L = c(T)$, $dM^2 = -\frac{4\det(T)}{c(T)^2}$; now the second assertion follows from the first by dropping all but one term. $\square$

Remark 3.8. *For $\Lambda \in \mathrm{Cl}_d(M)$, the analytic conductor of $\mathcal{AI}(\Lambda)$ is bounded by $|d|M^2$ [Roh94, (a2)]. As the analytic conductor of π_F is $\asymp k^2$, the convexity bound gives $L(1/2, \pi_F \times \mathcal{AI}(\Lambda)) \ll_\epsilon (k|d|M^2)^{1+\epsilon}$ and the Generalized Lindelöf hypothesis asserts that $L(1/2, \pi_F \times \mathcal{AI}(\Lambda)) \ll_\epsilon (k|d|M^2)^\epsilon$. On the other hand, lower bounds for $L(1, \pi_F, \mathrm{Ad})$ follow from good zero-free regions. In particular, it is known unconditionally (use [Bru06, Theorem 3] and functoriality) that there exists an absolute constant A such that $L(1, \pi_F, \mathrm{Ad}) \gg k^{-A}$; under GRH we have $L(1, \pi_F, \mathrm{Ad}) \gg_\epsilon k^{-\epsilon}$. It follows from the above discussion that under GRH we may take $\alpha = \beta = 0$ in (50).*

3.4. Sup-norms of Siegel cusp forms.

3.4.1. *Preparatory lemmas.* The results in this subsection are implicitly or explicitly contained in [Blo15]. Let $Y = \begin{bmatrix} x & \frac{y}{2} \\ \frac{y}{2} & z \end{bmatrix} \in M_2(\mathbb{R})$. We say that Y is Minkowski-reduced if the following holds

$$|y| \le x \le z, \qquad x \gg 1, \qquad \det Y \asymp xz.$$

Let $T = \begin{bmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{bmatrix} \in \Lambda_2$. Then TY is diagonalizable over $\mathbb{R}$ with positive eigenvalues; we shall denote them by x_1, x_2. We need some control over the counting function

$$N_Y(\lambda, h) = \#\{T \in \Lambda_2 : \lambda - h \le x_1, x_2 \le \lambda + h\}.$$

The first lemma provides a trivial bound.

Lemma 3.9. *Assume that Y is Minkowski-reduced. Then for $\lambda > 0$ and $h > 1$ we have*

$$N_Y(\lambda, h) \ll \frac{h^2 \lambda}{(\det Y)^{\frac{3}{2}}}.$$

Proof. For convenience, define $m = \lambda - h$ and $M = \lambda + h$. Then x_1, x_2 are the roots of the polynomial $X^2 - \mathrm{Tr}(TY)X + \det(TY)$. In particular, if $m \le x_1, x_2 \le M$ then it follows

$$(51) \qquad 2m \le x_1 + x_2 = \mathrm{Tr}(TY) = ax + \frac{1}{2}by + cz \le 2M$$

and

$$(52) \qquad m^2 \le x_1 x_2 = \det(TY) \le M^2.$$

From (51) we obtain

$$(53) \qquad 2m - \frac{1}{2}by - cz \le ax \le 2M - \frac{1}{2}by - cz.$$

Since $c > 0$, substituting (53) in (52), and using that Y is Minkowski-reduced, we get

$$c\left(2M - \frac{b}{2}y - cz\right) - \frac{x}{4}b^2 \ge \frac{\det(T)x}{4} \gg \frac{m^2}{z}.$$

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Changing variables $\tilde{c} = c - \frac{xM}{\det Y}$ and $\tilde{b} = b + \frac{yM}{\det Y}$ we obtain

(54)
$$z\tilde{c}^2 + \frac{x}{4}\tilde{b}^2 + \frac{y}{2}\tilde{b}\tilde{c} \ll \frac{xM^2}{\det Y} - \frac{m^2}{z} \asymp \frac{M^2 - m^2}{z}.$$

Completing the square,

$$z\left(\tilde{c} + \frac{y}{4z}\tilde{b}\right)^2 + \frac{\det Y}{4z}\tilde{b}^2 \ll \frac{M^2 - m^2}{z}$$

and thus $\left|\tilde{c} + \frac{y}{4z}\tilde{b}\right| \leq \frac{\sqrt{M^2 - m^2}}{z}$, $|\tilde{b}| \leq 2\sqrt{\frac{M^2 - m^2}{\det(Y)}}$. So there are $\ll \frac{M^2 - m^2}{z\sqrt{\det Y}}$ choices for the pair (b, c). For each choice of (b, c), by (53) there are $\ll \frac{M - m}{x}$ choices for a, giving the desired bound. $\qquad\square$

Lemma 3.10 (Blomer, Lemma 4 of [Blo15])**.** *Assume that Y is Minkowski-reduced and that $h = O(\sqrt{k}\log k)$. Then*

$$N_Y\left(\frac{k}{4\pi}, h\right) \ll \frac{k^{\frac{3}{2}+\epsilon}}{(\det Y)^{\frac{3}{4}}}.$$

For $x > 0$ and $k > 0$ define

$$g_k(x) = x^{\frac{k}{2}}\exp(-2\pi x), \quad x_k = \frac{k}{4\pi}.$$

Lemma 3.11. *The function g_k reaches its maximum at x_k, and for $k \gg 1$ we have*

(1) if $|x - x_k| \geq \sqrt{k}\log(k)$ then $g_k(x) \ll k^{-100}g_k(x_k)$,
(2) if $x \geq 2x_k$ then $g_k(x) \leq \exp(-\frac{\pi}{3}(x - x_k))g_k(x_k)$.

Proof. The derivative of $\log(g_k)$ is given by $\frac{d}{dx}\log g_k(x) = 2\pi(\frac{x_k}{x} - 1)$, thus g_k increases from 0 to x_k and decreases after this point, which establishes the first claim. To show (1), it suffices to bound $g_k(x_k \pm \sqrt{k}\log(k))$. If $x = x_k - \sqrt{k}\log(k)$ then we have

$$\frac{g_k(x)}{g_k(x_k)} = \exp\left(\frac{k}{2}\log\left(1 - 4\pi\frac{\log(k)}{\sqrt{k}}\right) + 2\pi\sqrt{k}\log(k)\right)$$
$$\leq \exp\left(-4\pi^2(\log k)^2\right).$$

On the other hand if $x \geq x_k$ then we have

$$\log g_k(x) = \log\left(g_k\left(\frac{x + x_k}{2}\right)\right) + 2\pi\int_{\frac{x+x_k}{2}}^{x}\left(\frac{x_k}{t} - 1\right) dt$$
$$\leq \log(g_k(x_k)) + 2\pi\int_{\frac{x+x_k}{2}}^{x}\frac{x_k - x}{x + x_k} dt$$
$$= \log(g_k(x_k)) - \pi\frac{(x - x_k)^2}{x + x_k}.$$

Thus

(55)
$$g_k(x) \leq g_k(x_k)\exp\left(-\pi\frac{(x - x_k)^2}{x + x_k}\right).$$

In particular, for $x = x_k + \sqrt{k}\log(k)$, by (55) and using the fact that $\sqrt{k}\log(k) \leq \frac{2}{e}k$, we have

$$g_k(x) \leq g_k(x_k)\exp\left(-\frac{4\pi^2 e}{e + 8\pi}(\log k)^2\right).$$

This proves (1). Finally, if $x \geq 2x_k$ then $\frac{x - x_k}{x + x_k} \geq \frac{1}{3}$ hence (55) gives (2). $\qquad\square$

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3.4.2. *Main result.*

Theorem 3.12. *Let $F \in S_k(\Gamma)$ be a Hecke eigenform that is not a Saito–Kurokawa lift and normalized so that $\|F\|_2 = 1$. Let $0 \le \alpha$ and $0 \le \beta \le 1$, be constants such that the bound* (50) *holds for all $\Lambda \in \mathrm{Cl}_d(M)$ for all fundamental discriminants d and positive integers M. Then for all $Z = X + iY \in \mathbb{H}_2$ we have*

$$|(\det Y)^{\frac{k}{2}} F(Z)| \ll_\epsilon k^{\frac{5}{4}+\beta+\frac{\alpha}{2}+\epsilon}.$$

Proof. Write the Fourier expansion of F and apply Corollary 3.7:

$$(\det Y)^{\frac{k}{2}} F(Z) = (\det Y)^{\frac{k}{2}} \sum_{T \in \Lambda_2} a(F,T) e^{2\pi i \mathrm{Tr}(TZ)}$$

$$\ll_\epsilon \frac{(4\pi)^k}{\Gamma(k)} k^{\frac{1}{4}+\frac{\alpha}{2}+\epsilon} \sum_{T \in \Lambda_2} \frac{c(T)^{-\frac{1}{2}-\beta}}{\det(T)^{\frac{1}{2}-\frac{\beta}{2}-\epsilon}} \det(TY)^{\frac{k}{2}} e^{-2\pi \mathrm{Tr}(TY)}.$$

By Stirling's formula, we have

$$(56) \qquad\qquad \frac{(4\pi)^k}{\Gamma(k)} \asymp k^{\frac{1}{2}} \left(\frac{4\pi e}{k}\right)^k.$$

We then proceed as in [Blo15]: the matrix $X = TY$ is diagonalizable with positive eigenvalues x_1, x_2, and $\det(X)^{\frac{k}{2}} e^{-2\pi \mathrm{Tr}(X)} = g_k(x_1) g_k(x_2)$. Let $\mathcal{X}_0$ be the set of matrices $T \in \Lambda_2$ such that $\frac{k}{4\pi} - \sqrt{k} \log(k) \le x_1, x_2 \le \frac{k}{4\pi} + \sqrt{k} \log(k)$. For those matrices $T \in \mathcal{X}_0$, we have $\det(T) \asymp k^2 (\det Y)^{-1}$ and thus by (56) and bounding $c(T) \ge 1$ trivially we have

$$\frac{(4\pi)^k}{\Gamma(k)} k^{\frac{1}{4}+\frac{\alpha}{2}+\epsilon} \frac{c(T)^{-\frac{1}{2}-\beta}}{\det(T)^{\frac{1}{2}-\frac{\beta}{2}-\epsilon}} \det(TY)^{\frac{k}{2}} e^{-2\pi \mathrm{Tr}(TY)} \ll (\det Y)^{\frac{1}{2}-\frac{\beta}{2}} k^{-\frac{1}{4}+\beta+\frac{\alpha}{2}+\epsilon}.$$

Furthermore, by Lemma 3.10, assuming (without loss of generality) that Y is Minkowski-reduced, we have $\#\mathcal{X}_0 \ll k^{\frac{3}{2}+\epsilon} (\det Y)^{-\frac{3}{4}}$ such matrices T, and thus we get a contribution

$$(\det Y)^{\frac{k}{2}} \sum_{T \in \mathcal{X}_0} a(F,T) e^{2\pi i \mathrm{Tr}(TZ)} \ll_\epsilon (\det Y)^{-\frac{1}{4}-\frac{\beta}{2}} k^{\frac{5}{4}+\beta+\frac{\alpha}{2}+\epsilon}.$$

Now for $j > 0$, let $\mathcal{X}_j$ be the set of matrices $T \in \Lambda_2$ such that $0 < x_1, x_2 \le (j+1)\frac{k}{4\pi}$ and $T \notin \bigcup_{i=0}^{j-1} \mathcal{X}_i$. Then combining Lemma 3.9, part (1) of Lemma 3.11 and (56) we get

$$(\det Y)^{\frac{k}{2}} \sum_{T \in \mathcal{X}_1} a(F,T) e^{2\pi i \mathrm{Tr}(TZ)} \ll k^{-50} (\det Y)^{-\frac{3}{2}}.$$

Finally, for $j \ge 2$, combining Lemma 3.9, part (2) of Lemma 3.11 and (56) we get

$$(\det Y)^{\frac{k}{2}} \sum_{T \in \mathcal{X}_j} a(F,T) e^{2\pi i \mathrm{Tr}(TZ)} \ll \exp\left(-\frac{jk}{13}\right) (\det Y)^{-\frac{3}{2}},$$

and thus

$$(\det Y)^{\frac{k}{2}} F(Z) = (\det Y)^{\frac{k}{2}} \sum_{j \ge 0} \sum_{T \in \mathcal{X}_j} a(F,T) e^{2\pi i \mathrm{Tr}(TZ)} \ll_\epsilon (\det Y)^{-\frac{1}{4}-\frac{\beta}{2}-\epsilon} k^{\frac{5}{4}+\beta+\frac{\alpha}{2}+\epsilon} \ll k^{\frac{5}{4}+\beta+\frac{\alpha}{2}+\epsilon}.$$

$\square$

References

[AS01]   Mahdi Asgari and Ralf Schmidt. Siegel modular forms and representations. *Manuscripta Math.*, 104(2):173–200, 2001.

[AZ95]   A. N. Andrianov and V. G. Zhuravlëv. *Modular forms and Hecke operators*, volume 145 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1995. Translated from the 1990 Russian original by Neal Koblitz.

[BHMM20]   Valentin Blomer, Gergely Harcos, Péter Maga, and Djordje Milićević. The sup-norm problem for GL(2) over number fields. *J. Eur. Math. Soc. (JEMS)*, 22(1):1–53, 2020.

[Blo15]   Valentin Blomer. On the size of Ikeda lifts. *Manuscripta Math.*, 148(3-4):341–349, 2015.

[BP16]   Valentin Blomer and Anke Pohl. The sup-norm problem on the Siegel modular space of rank two. *Amer. J. Math.*, 138(4):999–1027, 2016.

[Bru06]   Farrell Brumley. Effective multiplicity one on GL_N and narrow zero-free regions for Rankin-Selberg L-functions. *Amer. J. Math.*, 128(6):1455–1474, 2006.

[BT20]   Farrell Brumley and Nicolas Templier. Large values of cusp forms on GL_n. *Selecta Math. (N.S.)*, 26(4):Paper No. 63, 71, 2020.

[DK22]   Soumya Das and Hariram Krishna. Bounds for the Bergman kernel and the sup-norm of holomorphic Siegel cusp forms. *arXiv*, 2022.

[DPSS20]   Martin Dickson, Ameya Pitale, Abhishek Saha, and Ralf Schmidt. Explicit refinements of Böcherer's conjecture for Siegel modular forms of squarefree level. *J. Math. Soc. Japan*, 72(1):251–301, 2020.

[FM22]   Masaaki Furusawa and Kazuki Morimoto. On the Gross-Prasad conjecture with its refinement for $(SO(5), SO(2))$ and the generalized Böcherer conjecture. *arXiv*, 2022.

[Fre91]   Eberhard Freitag. *Singular modular forms and theta relations*, volume 1487 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1991.

[Fur93]   Masaaki Furusawa. On L-functions for $GSp(4) \times GL(2)$ and their special values. *J. Reine Angew. Math.*, 438:187–218, 1993.

[HL94]   Jeffrey Hoffstein and Paul Lockhart. Coefficients of Maass forms and the Siegel zero. *Ann. of Math. (2)*, 140(1):161–181, 1994. With an appendix by Dorian Goldfeld, Hoffstein and Daniel Lieman.

[HS20]   Yueke Hu and Abhishek Saha. Sup-norms of eigenfunctions in the level aspect for compact arithmetic surfaces, II: newforms and subconvexity. *Compositio Math.*, 156(11):2368–2398, 2020.

[HT13]   Gergely Harcos and Nicolas Templier. On the sup-norm of Maass cusp forms of large level. III. *Math. Ann.*, 356(1):209–216, 2013.

[IS95]   H. Iwaniec and P. Sarnak. L^{∞} norms of eigenfunctions of arithmetic surfaces. *Ann. of Math. (2)*, 141(2):301–320, 1995.

[JLS23]   Jesse Jääsaari, Stephen Lester, and Abhishek Saha. On fundamental Fourier coefficients of Siegel cusp forms of degree 2. *J. Inst. Math. Jussieu*, 22(4):1819–1869, 2023.

[KNS22]   Ilya Khayutin, Paul D. Nelson, and Raphael S. Steiner. Theta functions, fourth moments of eigenforms, and the sup-norm problem II. *arXiv*, 2022.

[KS22]   Ilya Khayutin and Raphael S. Steiner. Theta functions, fourth moments of eigenforms, and the sup-norm problem I. *arXiv*, 2022.

[KST12]   Emmanuel Kowalski, Abhishek Saha, and Jacob Tsimerman. Local spectral equidistribution for Siegel modular forms and applications. *Compositio Math.*, 148(2):335–384, 2012.

[Lap03]   Erez M. Lapid. On the nonnegativity of Rankin-Selberg L-functions at the center of symmetry. *Int. Math. Res. Not.*, 2003(2):65–75, 2003.

[Liu16]   Yifeng Liu. Refined global Gan-Gross-Prasad conjecture for Bessel periods. *J. Reine Angew. Math.*, 717:133–194, 2016.

[Mar14]   Simon Marshall. Sup norms of Maass forms on semisimple groups. *arXiv*, 2014.

[Mar21]   Jolanta Marzec. Maass relations for Saito-Kurokawa lifts of higher levels. *Ramanujan J.*, 55(1):25–51, 2021.

[NPS13]   Hiro-aki Narita, Ameya Pitale, and Ralf Schmidt. Irreducibility criteria for local and global representations. *Proc. Amer. Math. Soc.*, 141(1):55–63, 2013.

[PS09]   Ameya Pitale and Ralf Schmidt. Integral representation for L-functions for $GSp_4 \times GL_2$. *J. Number Theory*, 129(6):1272–1324, 2009.

[PS14]   Ameya Pitale and Ralf Schmidt. Bessel models for GSp(4): Siegel vectors of square-free level. *J. Number Theory*, 136:134–164, 2014.

[PS22]   Biplab Paul and Abhishek Saha. On Fourier coefficients and Hecke eigenvalues of Siegel cusp forms of degree 2. *International Mathematics Research Notices*, 2022.

[PSS17]   Ameya Pitale, Abhishek Saha, and Ralf Schmidt. Local and global Maass relations. *Math. Z.*, 287(1-2):655–677, 2017.

[Qiu21]   Yannan Qiu. The Bessel period functional on SO(5): The nontempered case. *arXiv*, 2021.

[Roh94]   David Rohrlich. Elliptic curves and the Weil-Deligne group. In *Elliptic curves and related topics*, volume 4 of *CRM Proc. Lecture Notes*, pages 125–157. Amer. Math. Soc., Providence, RI, 1994.

[RS74]   Howard Resnikoff and R. L. Saldana. Some properties of Fourier coefficients of Eisenstein series of degree two. *J. Reine Angew. Math.*, 265:90–109, 1974.

[RS07]   Brooks Roberts and Ralf Schmidt. *Local newforms for GSp(4)*, volume 1918 of *Lecture Notes in Mathematics*. Springer, Berlin, 2007.

[Sah17]   Abhishek Saha. Hybrid sup-norm bounds for Maass newforms of powerful level. *Algebra and Number Theory*, 11(5):1009–1045, 2017.

[Sah20]   Abhishek Saha. Sup-norms of eigenfunctions in the level aspect for compact arithmetic surfaces. *Math. Ann.*, 376(1-2):609–644, 2020.

[Ste20]   Raphael S. Steiner. Sup-norm of Hecke-Laplace eigenforms on S^3. *Math. Ann.*, 377(1-2):543–553, 2020.

[Sug85]   Takashi Sugano. On holomorphic cusp forms on quaternion unitary groups of degree 2. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 31(3):521–568, 1985.

[Xia07]   Honggang Xia. On L^∞ norms of holomorphic cusp forms. *J. Number Theory*, 124(2):325–327, 2007.