On the change of variables $\lambda \mapsto \sqrt{\lambda}$

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Asymptotic Geometric Analysis 2019.
Celebrating Vitali Milman's 80th birthday.

Introduction

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[. . .] the program is impressive and definitely worth to spend time on it. And if you feel the progress then it is OK to continue. I am always worry on a situation of no progress. Because this situation may continue infinite time.
Just I think (again) that if the largest goals are not moving, think where you may reduce goals but to receive the results to the end. This is usually important not only for self-satisfaction and (as Jean said) not to feel himself "an impotent", but also it organize correctly a piece and "free" our brain preparing it to the next step.
You know all this my philosophy, but one should also use it.

(15.1.2008)

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- $\|\phi_f^{(2k)}\|$ grows roughly as $\|f^{(k)}\|$

I. (Simple) analytic quasianalyticity

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Any function $f(z) = \sum_{n=0}^{\infty} a_n z^n \not\equiv 0$ with $|a_n| \leq e^{-\epsilon\sqrt{n}}$ (for some $\epsilon > 0$) has a finite number of zeros in $\overline{\mathbb{D}}$, counting multiplicity.

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i.e. ϕ_f admits an analytic extension to a strip (of width ϵ/C). $\square$

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Application: non-selfadjoint Schroedinger operator

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рассматривается при краевых условиях

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Consider the relative determinant "$\frac{\det(L-\lambda)}{\det(L_0-\lambda)}$", with zeros at the eigenvalues of L, as a function of $z = \frac{\lambda-i}{\lambda+i}$. It is of the form $f(z) = \sum_{n\geq 0} a_n z^n$ with a_n decaying roughly as $q(x)$, i.e. as $e^{-\epsilon\sqrt{n}}$.
Hence it has a finite number of zeros! $\square$

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and hence of the full Pavlov theorem, i.e. instead of $\int_0^\infty e^{\epsilon\sqrt{x}}|q(x)|dx < \infty$ one may assume $\int_0^\infty W(x)|q(x)|dx < \infty$ as long as $\left[\int^{\infty} \frac{\log W(x)}{x^{3/2}} dx = \infty\right]$ + regularity.

II. Non-symmetric quasianalyticity (Volberg $\sim$'80)

Definition

$(W_n \in [1, \infty])_{n \in \mathbb{Z}}$ is quasianalytic if $f(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta} \not\equiv 0$ with $\sup |a_n| W_n < \infty$ can not have a zero of infinite order (on $\mathbb{T}$).

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  M. Sodin '96: $\begin{cases} \text{optimal cond. on } W|_{\mathbb{Z}_+} \text{ ("normal QA")} \\ \sum_{n<0} \frac{\log W_n}{1+n^2} = \infty + \text{REG.}^- \text{ on } W|_{\mathbb{Z}_-} \end{cases}$

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Vul '59[(+)]: Let $p : \mathbb{R}_+ \to \mathbb{R}_+$ be non-decreasing, convex. TFAE:

$$\text{(a)} \quad \sigma \mapsto \int_{-\infty}^{\infty} \cos(x\sqrt{\lambda}) d\sigma(\lambda)$$

is injective on the class of measures

$$\int_{-\infty}^{\lambda} |d\sigma(\lambda')| \leq C \exp(-p(\sqrt{|\lambda|})) \quad (\lambda < 0)$$

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A couple of questions

- *Minimal* regularity on $W|_{\mathbb{Z}_-}$ <u>and</u> $W|_{\mathbb{Z}_+}$?

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Thanks for your attention!