due: 22/12/15

Assignment #4

① Let $+\Delta_\Omega^N$ be the Neumann Laplacian for a domain $\Omega \subset \mathbb{R}^d$. For $\Omega = [0,1]^d$, prove that

$$\mathcal{N}(\lambda; -\Delta_\Omega^N) = \frac{|B_d|}{(2\pi)^d}\lambda^{d/2} + o(\lambda^{d/2}), \quad \lambda \to +\infty,$$

where $|B_d|$ is the volume of the unit ball.

② If Ω is a bounded domain with $\mathrm{mes}_d(\partial\Omega) = 0$,

$$\mathcal{N}(\lambda; -\Delta_\Omega^D) \geq \frac{|B_d|}{(2\pi)^d}\cdot|\Omega|\,\lambda^{d/2} + o(\lambda^{d/2}), \quad \lambda \to +\infty.$$

③ In $d=3$, $(-\Delta+\xi)^{-1}$ is an integral operator with kernel $\frac{1}{4\pi|x-y|}\exp(-\sqrt{\xi}\,|x-y|)$

④ Suppose Ω is a bounded domain with smooth boundary; let $(\lambda_n)_{n\geq 0}$ be the collection of eigenvalues of $-\Delta_\Omega^D$. Prove

$$\zeta_\Omega(s) = \sum \lambda_n^{-s} \quad \text{is analytic on } \{\mathrm{Re}\, s > \tfrac{3}{2}-\varepsilon,\ s \neq \tfrac{3}{2}\},$$

and ζ_Ω has a simple pole at $s = 3/2$.

⑤ Let $\sigma > 0$, and let μ be a measure on $\mathbb{R}_+$ such that

$$(1)\ \int_0^\infty e^{-s\lambda}\,d\mu(\lambda) < \infty,\ s>0, \quad (2)\ \int_0^\infty e^{-s\lambda}\,d\mu(\lambda) \sim A s^{-\sigma}, \quad s \to +\infty$$

Prove: $\mu[0,\lambda) \sim \dfrac{A}{\Gamma(\sigma+1)}\lambda^\sigma, \quad \lambda \to +0.$