Assignment #3

due 8/12/2015

① Fix $a>0$, and let

$$(H\psi)(n) = \psi(n-1) + \psi(n+1) + a(-1)^n \psi(n), \qquad \psi : \mathbb{Z} \to \mathbb{C}.$$

Find $\sigma(H)$, $\sigma_{ess}(H)$, $\sigma_d(H)$, and compute $\lim\limits_{n\to\infty} \frac{\log|\psi(n)|}{|n|}$ for eigenfunctions in the discrete spectrum when H acts

(a) on $\ell^2(\mathbb{Z})$

(b) on $\{\psi \in \ell^2(\mathbb{Z}) \mid \psi|_{\mathbb{Z}_-} \equiv 0\}$

② Let $H = -\hbar^2 \frac{d^2}{dx^2} + \frac{x^2}{4} + a x^4$, $a>0$, and let $\lambda_0(\hbar)$ be the ground state (lowest eigenvalue). Find $A \neq 0$, $\gamma > 1$ s.t.

$$\lambda_0(\hbar) = \frac{\hbar}{2} + A\hbar^\gamma + o(\hbar^\gamma), \qquad \hbar \to +0.$$

③ Let $|\xi\rangle = e^{-|\xi|^2/2} e^{\xi a^*} |0\rangle$, $\xi \in \mathbb{C}$,

where $a^* = \frac{1}{\sqrt{2}}\left(x - \frac{d}{dx}\right)$, $a = \frac{1}{\sqrt{2}}\left(x + \frac{d}{dx}\right)$, and $|0\rangle$ is the state $\pi^{-1/4}\exp(-x^2/2)$.

(a) $a|\xi\rangle = \xi|\xi\rangle$ (in particular, $|\xi\rangle$ is a minimal uncertainty state, cf HW 1)

(b) Compute $e^{-itH}|\xi\rangle$, where $H = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{x^2}{2}$.

④ Let $H = -\hbar^2 \frac{d^2}{dx^2} + \frac{1}{8}(x^2-1)^2$.

(a) For small $\hbar$, H has exactly two eigenvalues below $\hbar$.

(b) Let $\lambda_0 < \lambda_1$ be the eigenvalues from (a), and φ_0, φ_1 – the corresponding eigenvectors. Then $\lambda_1 - \lambda_0 = \inf\limits_{\substack{f:\, \varphi_0 f \perp \varphi_0 \\ f \neq 0}} \hbar^2 \frac{\int |\nabla f|^2 \varphi_0^2\, dx}{\int f^2 \varphi_0^2\, dx}$.

(c) $\limsup\limits_{\hbar\to+0} \hbar \log[\lambda_1(\hbar) - \lambda_0(\hbar)] \leq$

$$\leq \frac{-1}{\sqrt{8}} \int_{-1}^{1} (1-x^2)\, dx = \frac{-5}{6\sqrt{2}}.$$

[in fact, equality holds]