Assignment #2 due 24/11/2015

① Let $V \in C(\mathbb{R}^d)$, $V(x) \xrightarrow[|x| \to \infty]{} 0$. Prove:

(1) $-\Delta + V$ is essentially self-adjoint on C_0^∞

(2) $\sigma_{ess}(-\Delta + V) = [0, \infty)$

② Let A be a self-adjoint operator. There exists a compact operator K s.t. $\sigma_{ess}(A) = \sigma(A+K)$.

③ Suppose $V \in L^2_{loc}$, $V <_{Rellich} -\Delta$. Prove: If $H = -\Delta + V \geq a$, then C_0^∞ is a core for $\sqrt{H + a + 1}$.

④ Let $z, w \notin \sigma(A)$. If $K R_z[A]$ is compact, then so is $K R_w[A]$.

⑤ Let $(H\psi)(x) = \sum_{y \sim x} (\psi(x) - \psi(y))$, $\psi \in \ell^2(\mathbb{Z}^d)$, and let

$$\delta_0(x) = \begin{cases} 1, & x = 0 \\ 0 & \end{cases}$$

(a) compute the spectral measure of H w.r. to δ_0 (i.e. the measure μ_{δ_0, δ_0})

(b) Compute the spectral projections E_λ, $\lambda \in \mathbb{R}$

⑥ Let $H = -\Delta + V$ on $\mathbb{Z}^d$ ($-\Delta$ is the op from problem ⑤), and let $I \subset \mathbb{R}$ be an interval such that $\operatorname{dist}(I, \sigma(H) \setminus I) > 0$. Prove:

$$|P_I(x, y)| \leq C \exp(-\alpha \|x - y\|) \quad \text{for some } C, \alpha > 0$$

(here $P_I = \mathbb{1}_I(H)$ is the spectral projection).