Assignment #1

due: 10/11/2015

① Find all $\psi \in L^2(\mathbb{R}^d)$ for which all the inequalities
$(\langle\psi|X_j^2|\psi\rangle - \langle\psi|X_j|\psi\rangle^2)(\langle\psi|P_j^2|\psi\rangle - \langle\psi|P_j|\psi\rangle^2) \geq \frac{\hbar}{2}$
is saturated, where $P_j = -i\hbar\frac{\partial}{\partial x_j}$, X_j = mult. by x_j

② Prove: C_0^∞ is a core for $-\Delta$ on $\mathbb{R}^d$.

③ B is subordinate to A in the sense of Rellich iff
$\mathcal{D}_A \subset \mathcal{D}_B$, and $\exists \alpha \in (0,1)$, $\beta > 0$ so that
$\forall \psi \in \mathcal{D}_A \quad \|B\psi\|^2 \leq \alpha\|A\psi\|^2 + \beta\|\psi\|^2$.

④ If $d \geq 4$ and $V \in L^p(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ for some $p > d/2$, then V is subordinate to $-\Delta$ in the sense of Rellich.

⑤ Let $H = -\frac{\hbar^2}{2}\left[\frac{\Delta_1}{m_1} + \frac{\Delta_2}{m_2}\right] - \frac{e^2}{|x-y|}$, $m, \hbar, e^2 > 0$,
on $C_0^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$. Prove:
(a) H is essentially self-adjoint
(b) $H \simeq H_c \oplus H_{rel}$, where

$$H_c = -\frac{\hbar^2}{2}\frac{\Delta}{m_1+m_2}, \quad H_{rel} = -\frac{\hbar^2}{2}\frac{m_1+m_2}{m_1 m_2}\Delta - \frac{e^2}{|r|}.$$