Infrared bound

For $f_1, \cdots, f_n \in S(R)$, let

$$Z_n(f_1, \cdots, f_n) = \int \prod_{j=1}^{n} \{ds_j f_j(s_j) \exp(-(s_j - s_{j+1})^2)\}$$

where $s_{n+1} = s_1$.

Proposition 1. *If $f_1, \cdots, f_n \geq 0$, then*

$$Z_n(f_1, \cdots, f_n) \leq \prod_{j=1}^{n} Z_n(f_j, \cdots, f_j)^{1/n} \ .$$

The proof goes as follows. For $a > 0$ and $F, G \in S(\mathbb{R}^2)$, define

$$Q_a(F, G) = \int_{\mathbb{R}^2} d^2x \int_{\mathbb{R}^2} d^2y F(x) \bar{G}(y) \exp(-(x_1 - y_1)^2 - a(x_2 - y_2)^2) \ .$$

Lemma 1. $Q_a(F, F) \geq 0$.

Proof. Let $u_a(x) = \exp(-x_1^2 - a x_2^2)$, then $\hat{u}_a \geq 0$, hence

$$Q_a(F, F) = \langle F * u_a, F \rangle = \langle \hat{F} \hat{u}_a, \hat{F} \rangle = \int_{\mathbb{R}^2} d^2\xi |\hat{F}|^2 \hat{u}_a \geq 0 \ .$$

□

Obviously, $Q_a(G, F) = \overline{Q_a(F, G)}$. Hence Q_a is a positive semidefinite bilinear form on $S(\mathbb{R}^2)$.

Corollary 1. $|Q_a(F, G)| \leq \sqrt{Q_a(F, F) Q_a(G, G)}$.

Proof of Proposition 1. First consider the case $n = 2m$. Denote

$$F(x) = \int_{\mathbb{R}^{m-2}} \prod_{j=2}^{m-1} ds_j \prod_{j=1}^{m} f_j(s_j) \prod_{j=1}^{m-1} \exp(-(s_j - s_{j+1})^2) \ ,$$

where $s_1 = x_1$, $s_m = x_2$. Also set

$$G(x) = \int_{\mathbb{R}^{m-2}} \prod_{j=2}^{m-1} ds'_j \prod_{j=1}^{m} f_{m+j}(s'_j) \prod_{j=1}^{m-1} \exp(-(s'_j - s'_{j+1})^2) \ ,$$

where $s_1 = y_2$, $s_m = y_1$. Then

$$\begin{aligned} Z_{2m}(f_1, \cdots, f_{2m}) &= Q_1(F, G) \leq \sqrt{Q_1(F,F) Q_1(G,G)} \\ &= \sqrt{Z_{2m}(f_1, \cdots, f_m, f_m, \cdots, f_1) Z_{2m}(f_{2m}, \cdots, f_{m+1}, f_{m+1}, \cdots, f_{2m})} \ . \end{aligned}$$

Rotating each term and iterating, we obtain the claim.

Now consider $n = 2m - 1$. Denote

$$F(x) = \int_{\mathbb{R}^{m-2}} \prod_{j=2}^{m-1} ds_j \prod_{j=1}^{m-1} f_j(s_j) \sqrt{f_m(s_m)} \prod_{j=1}^{m-2} \exp(-(s_j - s_{j+1})^2) \ ,$$

where $s_1 = x_1$, $s_m = x_2$. Also set

$$G(x) = \int_{\mathbb{R}^{m-2}} \prod_{j=2}^{m-1} ds'_j \sqrt{f_m(s_1)} \prod_{j=2}^{m} f_{m-1+j}(s'_j) \prod_{j=1}^{m-1} \exp(-(s'_j - s'_{j+1})^2) \ ,$$

where $s_1 = y_2$, $s_m = y_1$. Then

$$Q_a(F, G) \leq \sqrt{Q_a(F,F) Q_a(G,G)} \ .$$

Letting $a \to \infty$, we obtain:

$$\begin{aligned} Z_{2m-1}(f_1, \cdots, f_{2m-1}) &\leq \sqrt{Z_{2m-1}(f_1, \cdots, f_{m-1}, f_m, f_{m-1} \cdots, f_1)} \\ &\times \sqrt{Z_{2m-1}(f_{2m-1}, \cdots, f_{m+1}, f_m, f_{m+1}, \cdots, f_{2m-1})} \ . \end{aligned}$$

Again, we rotate and interate. □

References

[1] J. Fröhlich, B. Simon, T. Spencer, Infrared bounds, phase transitions and continuous symmetry breaking. Comm. Math. Phys. 50 (1976), no. 1, 79–95.