Consider $W_N(\beta, x) = \mathrm{E}_x\left[e^{\sum_{n=1}^N \beta\omega(n,S_n) - N\beta^2/2}\right]$ for $x \in \mathbb{Z}^d$ with $d = 2$, where the $\omega(i, x)$ are assumed to be iid centered Gaussians and S_n is simple random walk. This is a directed polymer partition function. Take $\beta_N = \frac{\hat{\beta}}{\sqrt{R_N}}$ with $R_N = \mathrm{E}_0^{\otimes 2}\left[\sum_{n=1}^N \mathbf{1}_{S_n^1 = S_n^2}\right] \sim (\log N)/\pi$. Write $W_N = W_N(\beta_N, 0)$.

Caravena, Sun and Zygouras showed that for all $\hat{\beta} < 1$, $\log W_N \xrightarrow{(d)} \mathcal{N}\left(-\frac{\lambda^2}{2}, \lambda^2\right)$, with $\lambda^2(\hat{\beta}) = -\log(1-\hat{\beta}^2)$. They also showed that

$$\sqrt{R_N}\left(\log W_N(\beta_N, x\sqrt{N}) - \mathbb{E}\log W_N(\beta_N, x\sqrt{N})\right) \xrightarrow{(d)} \sqrt{\frac{\hat{\beta}^2}{1-\hat{\beta}^2}}G(x),$$

with $G(x)$ a log-correlated Gaussian field on $\mathbb{R}^2$. With an eye toward constructing multiplicative chaoses based on $\sqrt{R_N}\log W_N(\beta_N, x\sqrt{N})$, it is natural to consider the q-moments $r_{q,N} = \mathbb{E}W_N^q$, with $q \sim \sqrt{\log N}$.

Lygkonis and Zygouras have recently shown that for q finite independent of N, $r_{q,N}$ coincide with the exponential moments of Gaussians. In this talk, I will describe joint work with Clement Cosco, where we prove the following.

There exists $\hat{\beta}_0 \in (0,1)$ and $\alpha \in (0,1)$ such that for all $\hat{\beta} < \hat{\beta}_0$ and $q^2\lambda^2 \leq \alpha \log N$, one has:

$$r_{q,N} \leq e^{\binom{q}{2}\lambda^2 \frac{1}{1-r}(1+|\varepsilon_N|)},$$

with $r = 12\binom{q}{2}\frac{1}{\log N}\frac{\hat{\beta}^2}{1-\hat{\beta}^2} < 1$ and $\varepsilon_N = \varepsilon(N, \hat{\beta}) \to 0$ as $N \to \infty$. In particular, for all $\hat{\beta} < \hat{\beta}_0$, uniformly for $q^2 = o(\log N)$,

$$r_{q,N} \leq e^{\binom{q}{2}\lambda^2(1+o(1))}. \tag{1}$$

With the same method, we also prove that the estimate (1) holds for all $\hat{\beta} < 1$ at the cost of choosing $q^2 = o(\log N/\log\log N)$, giving (together with the Gaussian convergence of CSZ) an independent proof of the Lygkonis-Zygouras theorem.

I will discuss some background, the ideas of the proof, potential extensions, and open questions.