KdV Equation With Eergodic Initial Data

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CLASSICAL AND QUANTUM MOTION IN DISORDERED ENVIRONMENT
A random event in honour of Ilya Goldsheid's 70-th birthday
Queen Mary, University of London, 18-22/12/2017

Introduction

- KdV hierarchy $q = q(t, x)$

1st	shift	$\partial_t q = \partial_x q$
2nd	KdV eq.	$\boldsymbol{\partial_t q = \partial_x^3 q - 6q\partial_x q}$
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Known results

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3. **2016: Binder-Damanik-Goldstein-Lukic** global well-posedness for quasi periodic initial data
$q(x) = \sum_{k\in\mathbb{Z}^N} \widehat{f}(k)\, e^{ixk\cdot\alpha}$ with $\left|\widehat{f}(k)\right| \leq \epsilon e^{-\kappa_0|k|}$,
where $|k\cdot\alpha| \geq a_0 |k|^{-b_0}$, $0 < a_0 < 1$, $b_0 > N$

Step llike initial data

2011: A. Rybkin For some $\delta_\pm > 0$ let q be

$$q \in L^2\left(\boldsymbol{R}_+, e^{\delta_+|x|^{1/2}}dx\right), \ \ q \in L^2\left(\boldsymbol{R}_-, e^{-\delta_-|x|}dx\right)$$

and $\inf \mathrm{sp} L_q > -\infty$. If L_q has non-trivial 2fold a.c. spectrum, the solution to KdV with initial data q is given by

$$u(t,x) = -2\partial_x^2 \log\det\left(I + \mathbb{M}_{t,x}\right) \quad \text{for } t \geq 0,$$

where $\mathbb{M}_{t,x}$ is called Marchenko operator defined by

$$\mathbb{M}_{t,x} f(y) = \int_0^\infty M(t, y+s+2x) f(s)\, ds \quad \text{for } f \in L^2(\boldsymbol{R}_+)$$

with $M(t,y) = \sum_{n=1}^N c_n^2 e^{8\kappa_n^3 t} e^{-\kappa_n y} + \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^\infty e^{8i\lambda^3 t} e^{i\lambda y} R_+(\lambda)\, d\lambda,$

when $q \in L^1(\boldsymbol{R}, (1+|x|)\,dx)$. $\left\{-\kappa_n^2\right\}$ are the negative eigen-values of L_q, and $R_+(\lambda)$ is the right reflection coefficient.

Weyl m-function

- 1D Schrödinger op. on $\boldsymbol{R}$: $L_q = -\partial_x^2 + q$ for real valued $q \in L^1_{loc}(\boldsymbol{R})$ with $H(q)$ is essemtially self-adjoint on $L^2(\boldsymbol{R})$.

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- For $\forall z \in \boldsymbol{C} \backslash \boldsymbol{R}$, $\exists 1 f_\pm = f_\pm(x, z, q)$ satisfying

$$L_q f_\pm = z f_\pm, \text{ s.t. } f_\pm \in L^2(\boldsymbol{R}_\pm), \quad f_\pm(0) = 1$$

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- **Weyl m-functions: $m_\pm(z)$ holomorphic on $\boldsymbol{C} \backslash \boldsymbol{R}$**

$$m_\pm(z) = m_\pm(z, q) = \pm f_\pm'(0, z, q), \quad \left(m_\pm(z, q = 0) = i\sqrt{z}\right)$$

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- Define

$$m(z) = \begin{cases} -m_+\left(-z^2\right) & \text{if } \operatorname{Re} z > 0 \\ m_-\left(-z^2\right) & \text{if } \operatorname{Re} z < 0 \end{cases}.$$

m is holomorphic on $\boldsymbol{C} \backslash (\boldsymbol{R} \cup i\boldsymbol{R})$ and $\operatorname{Im} m(z) / \operatorname{Im} z > 0$.

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  m is holomorphic on $\boldsymbol{C} \backslash (\boldsymbol{R} \cup i\boldsymbol{R})$ and $\operatorname{Im} m(z) / \operatorname{Im} z > 0$.
- $\{m_\pm(z)\}$ are called **reflectionless** on $A \in \mathcal{B}(\boldsymbol{R})$ if

$$m_+(\lambda + i0) = -\overline{m_-(\lambda + i0)} \text{ a.e. on } A.$$

Main theorem

Let $\mathcal{Q}$ be the set of all q whose Weyl functions $m_\pm$ satisfy

$$m_\pm(-z) = \sqrt{z} + \sum_{k=1}^{n-1} a_k z^{-k+1/2} \pm \sum_{k=1}^{n-1} b_k z^{-k} + O\left(z^{-n}\right)$$

as $|z| \to \infty$ along C_α with real a_k, b_k for any $n \geq 1$, $\alpha > 0$, and $\inf \mathrm{sp} L_q > -\infty$. Set $e_x(z) = e^{xz}$

$$\Gamma = \left\{g;\ g = e^h \text{ with real odd polynomial } h\right\}.$$

Theorem

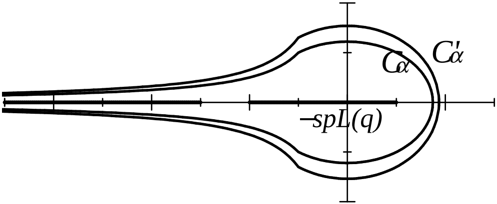
$\mathcal{Q} \subset C^\infty(\mathbf{R})$ holds, and $\tau_m(g) = \det(I + N_m(g))$ can be defined as a smooth function with respect to m, g, and $(K(g)q)(x) = -2\partial_x^2 \log \tau_m(ge_x)$ defines a flow on $\mathcal{Q}$. In particular

$$K(g_t)q(x) = \begin{cases} q(x+t) & \text{if } g_t(z) = e^{tz} = e_t(z) \\ \text{satisfies the KdV equation if } g_t(z) = e^{4tz^3} & \end{cases}.$$

Tau-function 1

- Assume $\mathrm{sp}L(q) > -\infty$ and let C_α, (C'_α) be a smooth curve surrounding $\mathrm{sp}L(q)$ such that for $x \geq 1$

$$C_\alpha = \left\{ z(x), \overline{z(x)} \right\}_{x \geq 0} \text{ with } z(x) = -x + ix^{-\alpha} \in \boldsymbol{C}_+.$$



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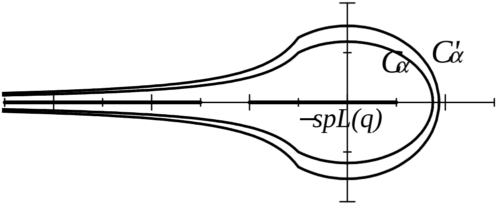

- Later α is chosen so that $g_e(z) = \left(g\left(\sqrt{z}\right) + g\left(-\sqrt{z}\right)\right)/2$, $g_o(z) = \left(g\left(\sqrt{z}\right) + g\left(-\sqrt{z}\right)\right) / \left(2\sqrt{z}\right)$ remain bounded on C_α.

$$g(z) = e^{z^3} \Longrightarrow g_e(z) = \cosh z^{3/2} \text{ and } \alpha = 1/2$$

Tau-function 2

- Define $\tau_m(g) = \det\left(I + N_m(g)\right)$ with integral operator $N_m(g)$ on $L^2\left(C_\alpha\right)$ with kernel

$$N_g\left(z, \lambda\right) = \frac{1}{2\pi i}\int_{C'_\alpha} \frac{\widehat{g}_o\left(\lambda'\right)\left(gm\right)_e\left(\lambda\right) + \widehat{g}_e\left(\lambda'\right)\left(gm\right)_o\left(\lambda\right)}{\left(\lambda' - z\right)\left(\lambda - \lambda'\right) m_o\left(\lambda'\right)} d\lambda'$$

where $\widehat{g}\left(z\right) = g(z)^{-1}$ and $g_e(z) = \left(g\left(\sqrt{z}\right) + g\left(-\sqrt{z}\right)\right)/2$, $g_o(z) = \left(g\left(\sqrt{z}\right) + g\left(-\sqrt{z}\right)\right) / \left(2\sqrt{z}\right)$.

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- $\tau_m(g)$ does not change by replacing m by $\widetilde{m}$ in M_g, where $\widetilde{m}(z) = m(z) - \delta(z)$ with δ_e, δ_o holomorphic in $\mathcal{D} \supset C'_\alpha$.
  Therefore, natural assumption: For any $n \geq 1$

$$\begin{cases} m_o(z) = 1 + \sum_{k=1}^{n-1} a_k z^{-k} + O\left(z^{-n}\right) & \text{as } |z| \to \infty \text{ along } C_\alpha \\ m_e(z) = \sum_{k=1}^{n-1} b_k z^{-k} + O\left(z^{-n}\right) & \text{as } |z| \to \infty \text{ along } C_\alpha \end{cases},$$

  under which one can show the traceability of $N_m(g)$ and $\tau_m(g) \neq 0$.

Sufficient conditions

Theorem

$\mathcal{Q}$ contains the classes of potentials below:
(i) $\mathcal{S}(\mathbb{R})$
(ii) Ergodic potentials having

$$\int_0^\infty \lambda^n \gamma(\lambda)\, d\lambda < \infty \quad \text{for any } n \geq 1,$$

which is satisfied when $q(x,\omega) \in C_b^\infty(\mathbf{R})$.
(iii) Smooth bounded potentials decaying sufficiently fast on one half axis and being ergodic on another axis.

Remark: If we are interested only in the KdV equation, we have only to assume the differentiability of initial functions q only up to a fixed number (≤ 16).

Ergodic initial data

Let $\mathcal{M}$ be the set of all ergodic probability measures on $\mathcal{Q}$. For $\mu \in \mathcal{M}$ and $g \in \Gamma$ one can define the induced measure $K(g)^{*}\mu$. Since, $K(g)$ commutes with the shift operation, we have

$$K(g)^{*}\mu \in \mathcal{M}.$$

Define the Floquet exponent w_{μ} by

$$w_{\mu}(z) = \mathbb{E}_{\mu}\left(m_{\pm}(z, q_{\omega})\right).$$

Then, the identities the IDS $N(\lambda) = \operatorname{Im} w_{\mu}(\lambda) / \pi$ and the Lyapunov exponent $\gamma(\lambda) = \operatorname{Re} w_{\mu}(\lambda)$ hold.

Theorem

$w_{\mu} = w_{K(g)^{*}\mu}.$

Proof 1

- Let $m_{\pm}$ be reflectionless on (λ_1, ∞), $\mathrm{infsp} L_q = \lambda_0 < 0$.

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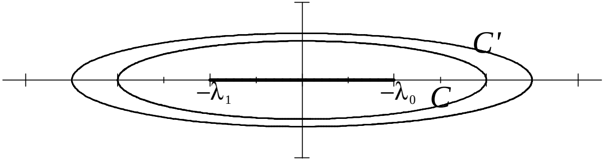

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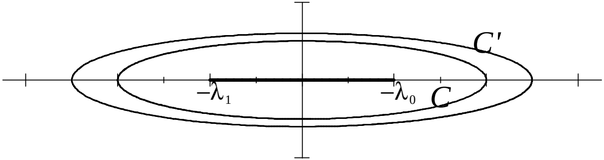

- For $g \in \Gamma$ let $N_m(g)$ be the operator on $L^2(C)$ with kernel

$$N_g(z, \lambda) = \frac{1}{2\pi i} \int_{C'} \frac{\widehat{g}_o(\lambda') (gm)_e(\lambda) + \widehat{g}_e(\lambda') (gm)_o(\lambda)}{(\lambda' - z)(\lambda - \lambda') m_o(\lambda')} d\lambda'.$$

$\tau_m(g) = \det(I + N_m(g))$ generates the KdV flow. This comes from Sato's theory developed by Segal-Wilson. The key in the proof is to factorize the Tau-function into two parts, one depends on $m_\pm$ and the other vanishes when taking the derivative twice.

Proof 2

- For ergodic potentials the property

$$m_{\pm}(-z) = \sqrt{z} + \sum_{k=1}^{n-1} a_k z^{-k+1/2} \pm \sum_{k=1}^{n-1} b_k z^{-k} + O\left(z^{-n}\right)$$

along C_α can be shown by $R(z)$ introduced by Rybkin

$$R(z) = \frac{m_+(z) + \overline{m_-(z)}}{m_+(z) + m_-(z)} \text{ and } \chi(z) = \frac{\gamma(z)}{\operatorname{Im} z} - \operatorname{Im} w'(z).$$

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- $4\chi\left(z\right)=\mathbb{E}\left(\left|R(z)\right|^{2}\left(\frac{1}{\operatorname{Im}m_{+}(z)}+\frac{1}{\operatorname{Im}m_{-}(z)}\right)\right)$

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- $4\chi\left(z\right) = \mathbb{E}\left(\left|R(z)\right|^2\left(\frac{1}{\operatorname{Im} m_+(z)} + \frac{1}{\operatorname{Im} m_-(z)}\right)\right)$
- $\mathbb{E}\left(\left|R(z)\right|\right) \leq \sqrt{2\chi\left(z\right)\operatorname{Im} w(z)}$

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- $4\chi(z) = \mathbb{E}\left(|R(z)|^2\left(\frac{1}{\operatorname{Im} m_+(z)} + \frac{1}{\operatorname{Im} m_-(z)}\right)\right)$
- $\mathbb{E}(|R(z)|) \leq \sqrt{2\chi(z) \operatorname{Im} w(z)}$
- $\xi_1 = \arg\left(-(m_+ + m_-)^{-1}\right)$,
  $\xi_2 = \arg m_+ m_- / (m_+ + m_-) \Longrightarrow$

$$\left|\xi_1 - \frac{\pi}{2}\right|, \quad \left|\xi_2 - \frac{\pi}{2}\right| \leq 2|R|.$$

Open problems

- Although one can construct a solution to the KdV equation with C^{∞} almost periodic initial data q, the almost periodicity of $K(g)q$ is not known. Sodin-Yuditski showed the almost periodicity if q is reflectionless on the spectrum Σ and Σ has a certain homogeneous property.

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- To obtain a solution to the KdV equation starting from very irregular initial data also remains open.
- Remling obtained a theorem on limit behavior of $K(e^{tz})q\,(x) = q\,(x+t)$ as $t \to \infty$. It is natural to expect a generalization of his theorem to $K(e^{th})q$ for general odd polynomial h.

Thank you for your attention !