Disordered Quantum-Spin Chains

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Interacting N-body Anderson model:

$$h_N = -\Delta + \lambda V_\omega(x) + \sum_{1 \le j < k \le N} U(x_j - x_k) \text{ in } \ell^2(\mathbb{Z}^{Nd})$$

$$V_\omega(x) = \sum_{j=1}^{N} \omega_{x_j}, \;\; x = (x_1, \ldots, x_N) \in \mathbb{Z}^{Nd}$$

Known: If $\lambda \geq \lambda_0(N) > 0$, then h_N is localized (Chulaevsky/Suhov, Aizenman/Warzel, Klein/Nguyen).

Problem: $\lambda_0(N) \to \infty$ as $N \to \infty$, localization not uniform in N, (similar issue with Lifshitz tail regime)

Physics question (e.g. Gornyi/Mirlin/Polyakov, Basko/Aleiner/Altshuler,...):

Are there regimes/variants of the interacting N-body Anderson model where suitable forms of localization hold **uniformly in** N (e.g. in the thermodynamic limit of an electron gas at positive particle density)?

"Many-body Localization (MBL)"

First question for mathematics (also for physics):

What is this???

Next thought:

Start with something easier!

Disordered quantum spin systems (chains):

$$H = \sum_{j\in\mathbb{Z}} h_{j,j+1} + \sum_{j\in\mathbb{Z}} t_j \quad \text{in} \quad \mathcal{H} = \bigotimes_{j\in\mathbb{Z}} \mathbb{C}^2$$

For simplicity:

- $h_{j,j+1}$ translation invariant interaction of spins at j and $j+1$
- t_j i.i.d. random 2×2-matrices acting on spin at j

Note: Single-particle Hilbert space is $\mathbb{C}^2$ (for spins) rather than $\ell^2(\mathbb{Z}^d)$ (as for d-dimensional "electrons"), so that single-particle physics becomes trivial

1st toy model: XY chain in random transversal field:

$$H_{XY} = \sum_j (\sigma_j^X \sigma_{j+1}^X + \sigma_j^Y \sigma_{j+1}^Y) + \sum_j \omega_j \sigma_j^Z$$

Can be used to start clarifying MBL phenomena:

Jordan-Wigner transform $\Longrightarrow$

$$H_{XY} \cong 2d\Gamma_a(h) + E_0 \quad \text{on } \mathcal{F}_a(\ell^2(\mathbb{Z}))$$

where h is the 1D Anderson model. **Free fermion system!**

Physically "trivial":

Anderson localization $\Longrightarrow$ (Full) many-body localization

MBL manifestations in random XY chain:

- Zero-velocity Lieb-Robinson bound for group/information transport (Hamza/Sims/St. 2012), **"Dynamical MBL"**
- Exponential decay of spatial correlations of all eigenstates and thermal states (Klein/Perez 1990, Sims/Warzel 2016)
- Area law for bipartite entanglement of all eigenstates, incl. dynamical entanglement (Pastur/Slavin 2014, Survey by Abdul-Rahman/Nachtergaele/Sims/St. 2017)

Proofs need to deal with (given Anderson localization):

Antisymmetry and **non-locality of Jordan-Wigner**

More challenging: Disordered XXZ (or XXX) chains

Physics (numerics): Expect MBL-transition* at low disorder (note that system is 1D).

Recent works by Beaud/Warzel, Elgart/Klein/St.:

Localization properties of the droplet spectrum in the Ising phase of the XXZ chain in random field

* Don't ask: We have nothing to say about the delocalized/thermalized phase...

The free XXZ chain:

$$H^0 = H^0_{XXZ} = -\frac{1}{4}\sum_j \left[\frac{1}{\Delta}(\sigma_j^X \sigma_{j+1}^X + \sigma_j^Y \sigma_{j+1}^Y) + (\sigma_j^Z \sigma_{j+1}^Z - 1)\right]$$

Assume **Ising phase**: $\Delta > 1$

True (but not so important for us): H^0 exactly diagonalizable via Bethe ansatz.

Important for us:

H^0 **preserves number of down-spins ("particles")**:

$$H^0 = \bigoplus_{N \geq 0} H^0_N$$

The N-particle operators:

$H_0^0 = 0$ on 1D space spanned by $|\ldots\uparrow\uparrow\uparrow\uparrow\ldots\rangle$ (vacuum)

$N \geq 1$:

$$H_N^0 \cong -\frac{1}{2\Delta} A_N + W \quad \text{on } \ell^2(\mathcal{X}^N),$$

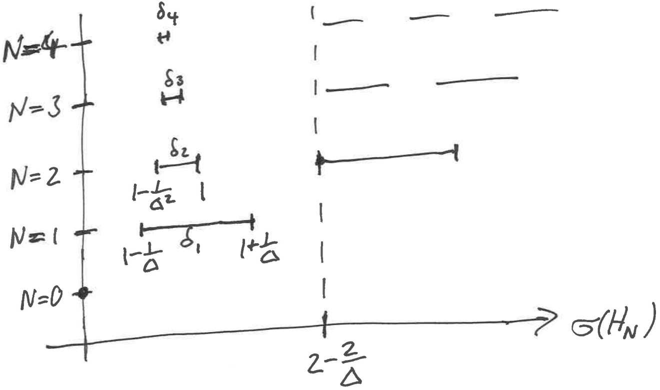
where

$$\begin{aligned}
\mathcal{X}^N &= \{x \in \mathbb{Z}^N : x_1 < x_2 < \ldots < x_N\} \quad \text{(down-spin sites)} \\
A_N &= \text{Adjacency operator on } \ell^2(\mathcal{X}^N) \\
W(x) &= \text{number of connected components of } (x_1, \ldots, x_N) \\
& \quad\ \text{(next neighbor attraction of hard core bosons)}
\end{aligned}$$

(i) W minimized for droplets (single cluster of down-spins), (ii) small hopping ($\Delta > 1$)

$\Longrightarrow$ Droplet regime at low energy (Nachtergaele/Starr 2002)

$\sigma(H_N)$ for general N



Droplet bands: (with $\cosh(\rho) = \Delta$)

$$\delta_N = \left[\tanh(\rho) \cdot \frac{\cosh(N\rho) - 1}{\sinh(N\rho)}, \tanh(\rho) \cdot \frac{\cosh(N\rho) + 1}{\sinh(N\rho)}\right]$$
$$\to \sqrt{1 - \frac{1}{\Delta^2}} \text{ as } N \to \infty$$

Droplet spectrum of H^0 (potentially including gap):

$$I_1 := \left[1 - \frac{1}{\Delta}, 2\left(1 - \frac{1}{\Delta}\right)\right)$$

Range of spectral projection $\chi_{I_1}(H^0)$ is **spanned** by states exponentially close to droplets. (Not fully localized, but close.)

Conjecture (suggested by B. Nachtergaele):

Adding disorder should fully localize the droplets, as these can be seen as a single quasi-particle in the one-dimensional XXZ model. Thus eigenstates to droplet spectrum should have only one "many-body localization center" (one cluster of downspins).

Recent rigorous proofs:

Beaud/Warzel 2017, Elgart/Klein/St. 2017

Infinite XXZ chain in random field:

$$H = H^0_{XXZ} + \lambda \sum_i \omega_j \mathcal{N}_j \quad \text{where } \mathcal{N}_j = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_j = \frac{1}{2}(I - \sigma_j^Z)$$

Assume: $\lambda > 0$ and

$$\omega_j \text{ i.i.d., } d\mu(\omega_j) = \rho(\omega_j)\, d\omega_j, \|\rho\|_\infty < \infty, \text{ supp}\, \rho = [0, \omega_{max}]$$

Finite volume chain on $\mathcal{H}^{(L)} = \bigotimes_{j=-L}^{L} \mathbb{C}^2$:

$$H^{(L)} = H^{0,[-L,L]}_{XXZ} + \lambda \sum_{j=-L}^{L} \omega_j \mathcal{N}_j + \beta(\mathcal{N}_{-L} + \mathcal{N}_L)$$

Assume: $\beta \geq \frac{1}{2}(1 - \frac{1}{\Delta})$ ("droplet b.c.", Nachtergaele/Starr)

Many-body eigencorrelator localization:

Theorem (Elgart/Klein/St. 2017)

Let $\delta > 0$, $\lambda > 0$ and $\Delta > 1$ be such that $\lambda\sqrt{\Delta - 1}$ is sufficiently large. Then there exist C and $m > 0$ such that

$$\mathbb{E}\left(\sum_{E\in\sigma(H^{(L)})\cap I_{1,\delta}} \|\mathcal{N}_j\psi_E\|\|\mathcal{N}_k\psi_E\|\right) \leq Ce^{-m|j-k|} \tag{1}$$

uniformly in $L > 0$, $j, k \in [-L, L]$.

Here ψ_E is the (almost surely unique) normalized eigenstate to $E \in \sigma(H^{(L)})$ and

$$I_{1,\delta} := \left[1 - \frac{1}{\Delta}, (2-\delta)\Big(1 - \frac{1}{\Delta}\Big)\right]$$

Remarks (instead of proof):

- Proof reduces to showing **uniform dynamical localization** (in N) of the operators

$$H_N = -\frac{1}{2\Delta} A_N + W(x) + \lambda \sum_{j=1}^{N} \omega_{x_j} \quad \text{in } \ell^2(\mathcal{X}^N)$$

- Crucial fact: (i) Attractive W-interaction, (ii) small degree of $\mathcal{X}^N$ at droplet configurations (uniform in N)
- Regime allows extension of known methods (here: Fractional moments method), works uniform in N
- IDS of H_N on $I_{1,\delta}$ decays exponentially in N (large deviations) $\Longrightarrow$ Summability in (1)

- **Higher energies?** Method of proof should extend to "k droplets" (i.e., MBL for $E \leq Ck$ if $\Delta \geq \Delta_0(k)$).

  Not good enough for physics! (They call our result "zero temperature localization" and really want $E \leq \rho L$ for MBL.)
- MBL of droplet spectrum for **more general geometries** (in preparation):

  Yes for quasi-1D systems (e.g. strips).

  No for higher dimensional systems (droplet band of H_0^N grows as $N^{(d-1)/d}$, the surface area of "ball" of volume N).
- Models without particle number preservation? General results for spin chains with large disorder? (Imbrie's work)

ICMP XIX, July 24–28, Montréal:

https://icmp2018.org/

Financial support available (in particular for junior researchers).

Happy Birthday, Ilya!