Harnack inequality for a degenerate random balanced operator

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I. Random media: microscopic $\leftarrow$ macroscopic

Macroscopic: Brownian motion

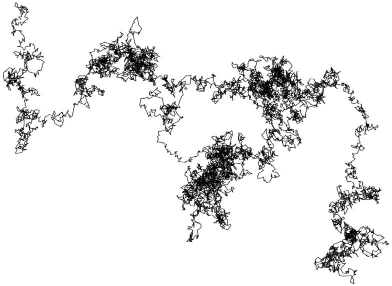

Figure: Brownian motion (from scratch.mit.edu/projects/143250914)

Microscopic: Random walks

Figure: Interstitial diffusion in a solid (from www.doitpoms.ac.uk)

I. **Q1** Brownian motion $\leftarrow$ RWRE

Macroscopic:

Brownian motion in $\mathbb{R}^d$ with **deterministic** "diffusivity matrix"

$$\bar{a} = (\bar{a}_{ij})_{1 \le i,j \le d} > 0.$$

Microscopic:

Markov chain in $\mathbb{Z}^d$ with **random** transition probabilities (called **environment/media**) which are possibly degenerate.

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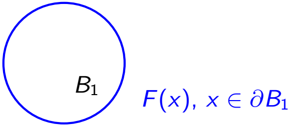
$$\text{Q1:} \quad B_t \Leftarrow X_{nt}/\sqrt{n} \quad ?$$

I. **Q2** PDE $\leftarrow$ Random difference equation

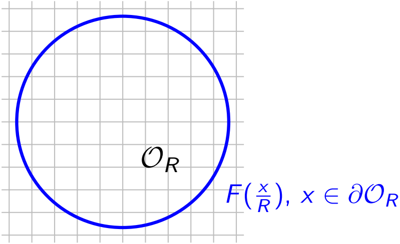
PDE in $\mathbb{R}^d$:



For **a deterministic** matrix $\bar{a} = (\bar{a}_{ij})$, consider solution v of the Dirichlet problem

$$\begin{cases} L_{\bar{a}} v(x) = 0 & \text{for } x \in B_1 \\ v(x) = F(x) & \text{on } \partial B_1, \end{cases}$$

Difference equation in $\mathbb{Z}^d$:



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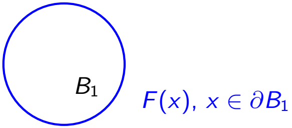

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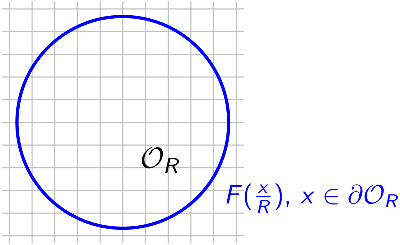

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Q2: $\quad v \Leftarrow u_{a,R} \quad$?

Assumptions

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Remarks Usually Assumption 2 is replaced by $P(a_i(x) > \lambda, \forall i) = 1$

- If $\lambda > 0$, this is called **uniformly-elliptic**. If $\lambda = 0$, **elliptic**.

Homogenization questions:

- There exists $\bar{a}$ such that for P-almost all $a = \{a(x) : x \in \mathbb{Z}^d\}$,

$$\lim_{R\to\infty} \max_{x\in\mathcal{O}_R} |u_{a,R}(x) - v(\tfrac{x}{R})| = 0?$$

- What is the rate of convergence

$$P\left(\max_{x\in\mathcal{O}_R} |u_{a,R}(x) - v(\tfrac{x}{R})| > \epsilon\right)?$$

II. Probablistic intepretation: RWRE

Let

$$a(x, x \pm e_i) := a_i(x).$$

This defines a random walk (X_n) with transition law denoted by P_a.

$$P_a(X_{n+1} = x \pm e_i | X_n = x) = a(x, x \pm e_i) = a_i(x).$$

Remark:

- $$\mathcal{L}_a u(x) = \sum_{y:y\sim x} a(x,y)[u(y) - u(x)].$$

  When $a(x, y) = a(y, x)$, this is a *divergence form* model.
- We call u a-**harmonic** if $\mathcal{L}_a u = 0$.
- Dynkin's formula.

II. Rigorous definition of RWRE **Q1**

Question about the random walk:

- Quenched Central Limit Theorem (QCLT)
  For P-almost every a,

$$\frac{X_{nt}}{\sqrt{n}} \Rightarrow \text{Brownian motion with deterministic covariance matrix } \bar{a}?$$

Related works

1. Non-divergence form
   (I.) Homogenization in uniformly-elliptic environment
   - Papanicolaou-Varadhan '80: homogenization
   - Yurinskii '80s: For $d \geq 5$, algebraic rate of homogenization
   - Caffarelli-Souganidis'10: sub-algebraic rate
   - Armstrong-Smart '14: stretch exponential rate

   (II.) QCLT:
   - Lawler '82 uniformly elliptic
   - G.-Zeitouni '12, Deuschel-G.-Ramirez'16, Berger-Deuschel-G.-Ramirez, Deuschel-G.'17
   - Berger-Desuchel '14: genuinely d-dimensional
2. Divergence form:
   (I.) Homogenization: Papanicolaou-Varadhan '80s Yurinskii '80s, Naddaf-Spencer 98, Gloria-Neukamm-Otto '14, Murrat, Armstrong-Smart

   (II.) QCLT: Kipnis-Varadhan, Sidoravicius-Sznitman'04 $d \geq 4$), Mathieu-Piatniski'07, Berger-Biskup '07, Armstrong-Dario-Murrat

   (III.) Harnack inequality (in a percolation cluster): Barlow '04

Our results

Theorem (homogenization)

For any $\epsilon > 0$, *there exist* $C = C(\epsilon, P)$ *and* $\delta = \delta(P)$ *such that*

$$P\left(\max_{x\in\mathcal{O}_R} |u_{a,R}(x) - v(\tfrac{x}{R})| > \epsilon\right) \le Ce^{-R^\delta}.$$

Main result:

Theorem (Harnack ineq.)

There exist constants $C = C(P)$, $\delta = \delta(P)$ *such that with probability at least* $1 - Ce^{-R^\delta}$, *for any non-negative a-harmonic function* $f : \mathcal{O}_{2R} \to \mathbb{R}$,

$$\max_{x\in\mathcal{O}_R} f \le C \min_{x\in\mathcal{O}_R} f.$$

III. Difficulties

1. local degeneracy
2. lack of connectivity (the most difficult part in our proof)
3. Covering argument will not work

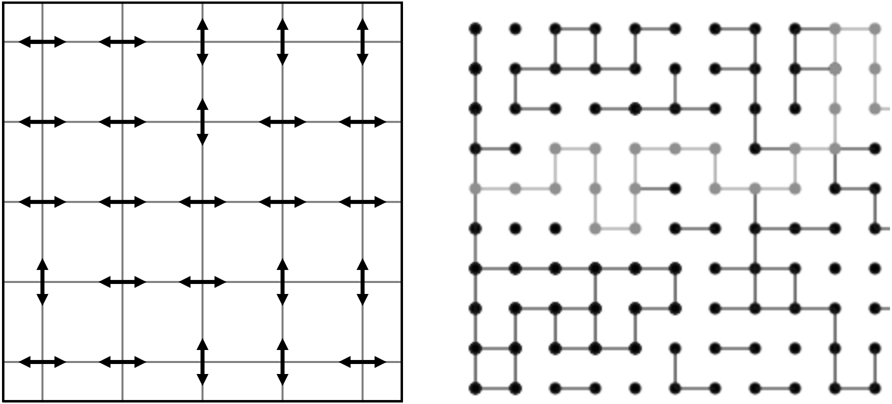

Figure: (Left) Our environment. (Right) The classical bond percolation.

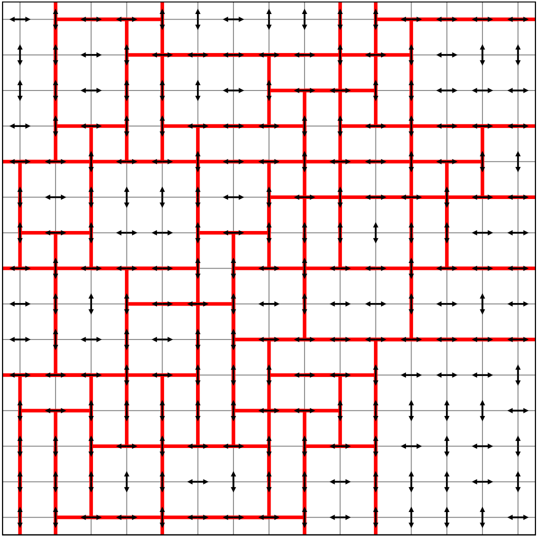

Figure: The "sink" in $\mathbb{Z}^2$.

Ingredients of our proof of the Harnack inequality

Theorem (Percolation)

$$P(\text{radius of 'holes'} \geq k) < Ce^{-ck^{\alpha}}.$$

Lemma (Oscillation estimate)

There exist constants $0 < \alpha < 1, C$ *such that with probability at least* $1 - Ce^{-R^{\delta}}$, *for any non-negative a-harmonic function* $f : \mathcal{O}_{2R} \to \mathbb{R}$,

$$\operatorname*{osc}_{\mathcal{O}_R} f \leq \alpha \operatorname*{osc}_{\mathcal{O}_{2R}} f.$$