Getting to the heart of random operators

An ideal wire is a perfect conductor. What about a material copper wire: how is the conductance affected by impurities? What about a copper plate? Researchers in the Spectrum project are studying these types of fundamental questions in theoretical physics from the mathematical perspective, as **Professor Sasha Sodin** explains

The motion of a quantum particle such as an electron is described by the Schrödinger equation. It was understood by P. W. Anderson in the 1950s that the presence of random impurities in the medium dramatically affects the properties of the solutions. The extreme situation is Anderson localisation, in which the impurities turn a conducting medium into an insulator.

Mathematically rigorous arguments appeared in the 1970s, when Goldsheid, Molchanov and Pastur proved that Anderson localisation always occurs in one-dimensional systems. Later, Fröhlich–Spencer and Aizenman–Molchanov proved that it occurs in an arbitrary dimension when the density of impurities is sufficiently high.

Many questions remain in this area however, which researchers in the EC-backed Spectrum project aim to investigate. “What happens when the density of impurities is low, and the dimension is greater than one, remains a mystery. Is a copper plate with impurities a conductor?” asks Professor Sasha Sodin, the project’s Principal Investigator. “The answer should depend on two key ingredients: the geometry of the problem and the randomness of the impurities.”

It is believed that a two-dimensional plate becomes an insulator at arbitrarily weak density of impurities, whereas a three-dimensional bar retains its conductance when the impurities are sufficiently sparse. “Presumably, the difference has to do with the different behaviour of classical random walk in two and three dimension. However, the connection between random walk and quantum dynamics in the presence of disorder is not well understood,” says Professor Sodin.

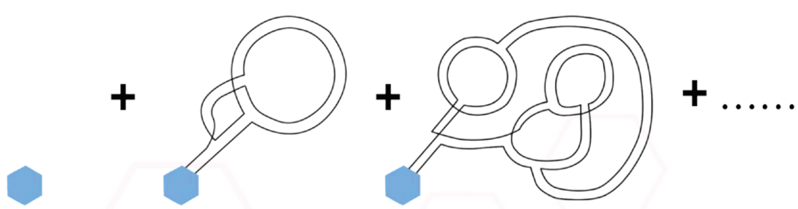
Some central questions remain open even for one-dimensional systems. The rigorous results asserting the absence of conductance in a wire with arbitrarily sparse impurities seems to run contrary to our everyday experience. “The results of Goldsheid–Molchanov–Pastur, along with most of the subsequent work, pertain to an infinitely long wire, whereas in reality the width of a wire is not negligible with respect to its length,” explains Professor Sodin. “Theoretical physicists, particularly Fyodorov and Mirlin, devised several approaches to this problem, but so far none of these has been mathematically justified.”

We aim to understand the **basic properties** of **quantum particles** moving through a **disordered environment**. We would like to know how the motion is affected by the combination of **randomness and geometry**

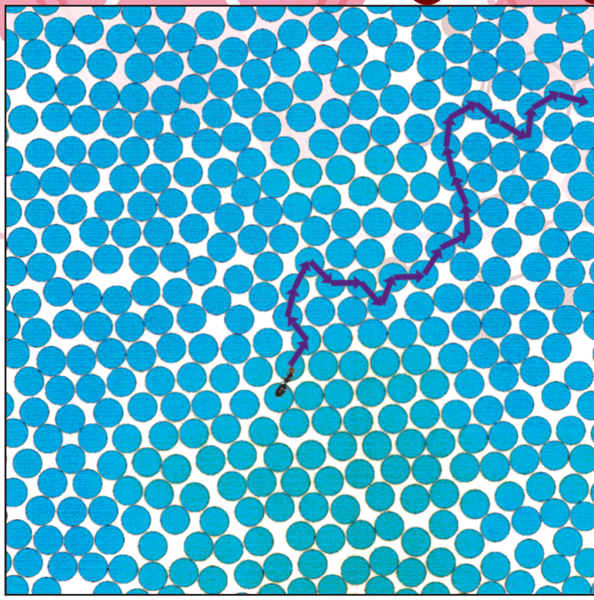
Resummation of divergent series

Physicists since Richard Feynman have used perturbation theory, in which the quantities of interest are expanded in an infinite series. The terms are labelled by graphs called Feynman diagrams, and the sum of the first few terms is considered to be an approximation to the exact solution. This procedure is very powerful; however, it suffers from serious drawbacks. “Typically, the answer given by perturbation theory is a divergent series, such as 1 – 3 + 9 – 27 + 81 - ...”, says Professor Sodin. “Such an expression has no mathematical meaning. By analogy with a geometric progression, one is tempted to conclude that the sum is ¼. However, it is not clear that such a guess gives the correct answer to the initial problem.”

Attaching a mathematical meaning to a divergent series is known as resummation.



Perturbative series in theoretical physics are often labelled by graphs called Feynmann diagrams.



The escape of an ant to infinity: in a typical 2D configuration of sufficiently high density, there is an infinite path avoiding gaps wider than 0.3mm. Figure courtesy of Prof. Dietrich Stoyan.

Different resummation procedures may lead to different sums, and some divergent expansions are not resummable. Therefore devising the right resummation procedure requires an understanding of the specific nature of that problem.

Offer Kopelevitch completed his MSc thesis at Tel Aviv University in 2016

Recently, Offer Kopelevitch, an MSc student at Tel Aviv University and a member of the Spectrum team, devised a rigorous resummation procedure, applicable to several spectral problems involving randomness. "This first step is extremely important," says Professor Sodin. "Previously, it was thought that there could be no mathematically consistent way to resum divergences of this kind. The next step is to tackle the problems with a stronger geometric component, such as conductance in a wire."

Gas of hard spheres

The famous Kepler conjecture, going back to the seventeenth century and recently proved by Hales, asserts that no arrangement of non-overlapping unit balls in 3D space can occupy more than $\pi/\sqrt{18} = 0.74...$ of the volume of the space. The bound is achieved for the so-called FCC packing, which is the one used by costermongers to arrange oranges in a box.

From the point of view of mathematical physics, it is important to understand the properties of a typical configuration chosen at random among the arrangements with a density below $\pi/\sqrt{18}$. "This model goes back to Ludwig Boltzmann, it is called a gas of hard spheres," comments Professor Sodin. "It is classical rather than quantum, however, the fascinating interplay between geometry and randomness makes it related to the other problems we study."

Typical configurations at low density are gas-like: moving the balls around in one region of space has very weak influence on the balls in distant regions. A mathematical proof was found by Ruelle in the 1960s. As the density gets closer to $\pi/\sqrt{18}$, it is expected but not proved that the system undergoes a phase transition, and typical configurations acquire similarity to a lattice.

Significant progress was made by Dr. Alexander Magazinov, a postdoctoral researcher at Tel Aviv University and a member of the Spectrum team, who answered a question posed by Bowen–Lyons–Radin–Winkler. "Suppose the balls are 1m wide, and an ant can jump between two balls if the distance between them is less than 1mm.," explains Professor Sodin. "The result of Magazinov asserts that, if the density is sufficiently close to $\pi/\sqrt{18}$, a typical configuration has an infinite cluster of balls such that an ant can reach any of them from any other one."

"Magazinov's result also holds in two dimensions, for configurations of disks, but not in one dimension. We do not know what happens in dimension four and above. The dimension-dependence is one of the things that made this problem so challenging."

At a glance

Full Project Title

Randomness and geometry in spectral theory and beyond it.

Project Objectives

Study the interplay of randomness and geometry in the spectral theory of random operators and in other parts of mathematical physics.

Project Funding

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Contact Details

QUEEN MARY UNIVERSITY OF LONDON
MILE END ROAD
E1 4NS
LONDON
United Kingdom
T: +44 20 7882 5452
E: a.sodin@qmul.ac.uk
W: http://www.maths.qmul.ac.uk/~s_sodin/

Professor Sasha Sodin

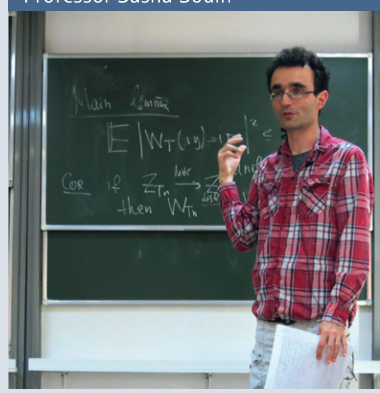

Photo courtesy of Professor A. Tikhomirov

Sasha Sodin is Professor of Analysis and Mathematical Physics at Queen Mary University London. He completed his PhD at Tel Aviv University in 2010, after which he worked at Princeton in the US, before taking up his current position in 2016. He has held a position in Tel Aviv University since 2014, from where he is on leave from 2016.

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