6 Stopping times and the first passage

Definition 6.1. Let $(\mathcal{F}_t, t \geq 0)$ be a filtration of σ-algebras. *Stopping time* is a random variable τ with values in $[0, \infty]$ and such that $\{\tau \leq t\} \in \mathcal{F}_t$ for $t \geq 0$.

We can think of stopping time τ as a strategy which at every time t decides to stop or not (until stopping decision is made) on base of the information available so far. For this reason τ is called a 'nonanticipating', 'online', or 'real-time' strategy. 'Markov time' is another name for stopping time. In financial mathematics stopping times may constitute a part of investor's policy. For instance, a holder of option of the American type faces the problem of choosing the moment to exercise the option.

The definition of τ says that the information available at time t determines if stopping has occured not later than t. But this implies that the decisiion to stop exactly at time t is also determined by the same information. Formally $\{\tau > t\} \in \mathcal{F}_t$, because the event is complementary to $\{\tau \leq t\}$. Observing that $\{\tau = t\} = \{\tau \leq t\} \cap (\cap_{k=1}^{\infty}\{\tau > t - 1/k\})$, we have here $\{\tau > t - 1/k\} \in \mathcal{F}_{t-1/k} \subset \mathcal{F}_t$ for every k. Now it follows from the properties of σ-algebras that also $\{\tau = t\} \in \mathcal{F}_t$.

Example 6.2. Our main example of stopping time is the *first passage time* for BM. For any 'level' x, define

$$\tau_x = \min\{t \geq 0 : B(t) = x\},$$

where $\min \varnothing = \infty$. This is a stopping time w.r.t. the natural filtration $(\mathcal{F}_t^B, t \geq 0)$ of the BM (or any larger filtration).

Let $M(t) := \max_{s \in [0,t]} B(s)$ be the 'running maximum' of BM on $[0, t]$. It is useful to note that for $x > 0$ by the continuity of BM we have

$$\{\tau_x \leq t\} = \{M(t) \geq x\}. \tag{35}$$

The relation actually proves that τ_x is a stopping time, since $M(t)$ is a function of $(B(s), s \in [0, t])$ (i.e. the random variable $M(t)$ is $\mathcal{F}_t^B$-measurable).

6.1 Stopped martingales

Let $(X(t), t \geq 0)$ be a random process, τ a stopping time. The *stopped process* is defined as $\widetilde{X}(t) = X(\tau \wedge t),\ t \geq 0$. On the event $\{\tau < \infty\}$ the stopped process becomes frozen at time τ, i.e. does not change value.

Theorem 6.3. *If $(X(t), t \geq 0)$ is a martingale, then the stopped process is also a martingale, whicnever the stopping time τ.*

Proof. We will give a complete proof under the assumption that τ assumes values in the countable set $\{0, 1, 2, \ldots, \infty\}$. We have

$$X(\tau \wedge n) = \sum_{k=0}^{n-1} X(k)1(\tau = k) + X(n)1(\tau = n) + X(n)1(\tau > n),$$

$$X(\tau \wedge (n+1)) = \sum_{k=0}^{n-1} X(k)1(\tau = k) + X(n)1(\tau = n) + X(n+1)1(\tau > n).$$

In the second formula $X(\tau \wedge (n+1)) = \sum_{k=0}^{n-1} X(k)1(\tau = k) + X(n)1(\tau = n)$ is $\mathcal{F}_n$-measurable, thus conditioning has no effect:

$$\mathbb{E}\left[\sum_{k=0}^{n-1} X(k)1(\tau = k) + X(n)1(\tau = n)|\mathcal{F}_n\right] = \sum_{k=0}^{n-1} X(k)1(\tau = k) + X(n)1(\tau > n).$$

From this

$$\mathbb{E}[X(\tau \wedge (n+1))|\mathcal{F}_n] - X(\tau \wedge n) = \mathbb{E}[X(n+1)1(\tau > n)|\mathcal{F}_n] - X(n)1(\tau > n) =$$
$$1(\tau > n)\mathbb{E}[X(n+1)|\mathcal{F}_n] - 1(\tau > n)X_n = 1(\tau > n)(\mathbb{E}[X(n+1)|\mathcal{F}_n] - X(n)) = 0,$$

where we used that $\{\tau > n\} \in \mathcal{F}_n$ and that X is a martingale.

The proof is literally the same for the case when τ takes values $\{0, 1/n, 2/n, \ldots, \infty\}$, where n is a fixed integer. The case of arbitrary τ follows by an approximation argument, which we omit. □

This result must be clear intuitively. The idea of a martingal is that of a gambler's capital in fair game. The theorem says that the game remains fair whichever nonanticipating strategy to exit the game.

For stopped martingale we have $\mathbb{E}X(\tau \wedge t) = \mathbb{E}X(0)$ for every t. Sending $t \to \infty$ we have $\tau \wedge t \to \infty$, so passing to limit one might expect that $\mathbb{E}X(\tau \wedge t) = \mathbb{E}X(0)$, but this is not always true.

Theorem 6.4. (Optional sampling theorem.) *Let martingale X and stopping time τ satisfy the conditions*

(i) *τ is finite a.s., that is $\mathbb{P}(\tau < \infty) = 1$,*

(ii) *$\mathbb{E}X(\tau) < \infty$,*

(iii) *$\lim_{t\to\infty} \mathbb{E}(X(t)1(\tau > t)) = 0$.*

Then $\mathbb{E}X(\tau) = \mathbb{E}X(0)$.

There are simpler conditions for $\mathbb{E}X(\tau) = \mathbb{E}X(0)$ to hold. It is enough to require that τ be bounded, that is $\mathbb{P}(\tau < K) = 1$ for some $K > 0$. Another sufficient condition is the uniform integrability of the martingale[6].

Example 6.5. Let $a, b \geq 0$. The first time the BM exits the interval $[-b, a]$ is $\tau = \min\{t : B(t) = a, \text{ or } B(t) = -b\}$. The exit through a (respectively $-b$) may be interpreted as ruin of a gambler playing a 'continuous head-or-tail' game with the initial capital a (respectively b). Let $p = \mathbb{P}(B(\tau) = a), q = \mathbb{P}(B(\tau) = -b)$. It can be shown that $p+q = 1$, that is $\mathbb{P}(\tau < \infty) = 1$. By the optional sampling theorem $0 = \mathbb{E}B(0) = \mathbb{E}B(\tau) = ap - bq$. Along with $p + q = 1$ we have

$$p = \frac{b}{a+b}, \quad q = \frac{a}{a+b},$$

[6]Random variables $X(t), t \geq 0$, are uniformly integrable if for every $\epsilon > 0$ there exists $K > 0$ such that $\mathbb{E}[|X(t)|\,1(|X(t)| > K)] < \epsilon$.

which is a classical fact.

Proving that $\tau < \infty$ is easy by looking at the distribution of $B(t)$ for large t. We leave details as an exercise.

Fix $x > 0$. The BM exits the interval $[-b, x]$ through point x with probability $b/(x+b)$. If the event occurs, the first passage time τ_x is finite. Letting $b \to \infty$,

$$\mathbb{P}(\tau_x < \infty) \geq \frac{b}{x+b} \to 1,$$

so τ_x is finite a.s. By symmetry this holds also for $x < 0$.

The optional sampling theorem has a kind of converse.

Theorem 6.6. *Suppose random process $(X(t), t \geq 0)$ satisfies $\mathbb{E}X(\tau) < \infty$ and $\mathbb{E}X(\tau) = \mathbb{E}X(0)$ for all bounded stopping times. The the process is a martingale.*

6.2 σ-algebra $\mathcal{F}_\tau$, martingales and strong Markov processes

Under mild conditions, the martingale and Markov properties of random processes can be strengthened by replacing fixed times by (random) stopping times.

Definition 6.7. Let τ be a stopping time w.r.t. filtration $(\mathcal{F}_t, t \geq 0)$. The σ-algebra $\mathcal{F}_\tau$ is the collection of events A such that $A \cap \{\tau \leq t\} \in \mathcal{F}_t$ for every $t \geq 0$.

If τ is thought of as a moment when observation of a random process stopped, $\mathcal{F}_\tau$ is interpreted as the collection of events observed before τ. For instance, $\{B(\tau/2) > 1\} \in \mathcal{F}_\tau^B$, but $\{B(\tau + 1) > 1\} \notin \mathcal{F}_\tau^B$.

Recall that the martingale property is $\mathbb{E}[X(t_2)|\mathcal{F}_{t_1}] = X(t_1), t_1 \leq t_2$ for fixed $t_1 \leq t_2$.

Theorem 6.8. *Let X be a uniformly integrable martingale and $\tau_1 \leq \tau_2 \leq \infty$ be two stopping times. Then*

$$\mathbb{E}[X(\tau_2)|\mathcal{F}_{\tau_1}] = X(\tau_1).$$

Recall further that $(X(t), t \geq 0)$ is a Markov process if for arbitrary fixed $t, s \geq 0$ holds $\mathbb{P}(X(t+s) \leq y|\mathcal{F}_t) = \mathbb{P}(X(t+s) \leq y|X(t))$ (for $y \in \mathbb{R}$).

Definition 6.9. Random process $(X(t), t \geq 0)$ adapted to filtration $(\mathcal{F}_t, t \geq 0)$ has the *strong Markov property* if for every finite stopping time τ and $s \geq 0$

$$\mathbb{P}(X(\tau + s) \leq y|\mathcal{F}_\tau) = \mathbb{P}(X(\tau + s) \leq y|X(\tau)).$$

Solutions to SDE's are strong Markov processes (i.e. have the strong Markov property).

In particular, the Brownian motion is a strong Markov process. For the BM the strong Markov property means that for every finite stopping time τ the process

$$\widetilde{B}(t) := B(\tau + t) - B(\tau), \quad t \geq 0$$

is a Brownian motion independent of $\mathcal{F}_\tau$.

The first passage time as a random process We may consider $(\tau_x, x \geq 0)$ as a random process, with parameter x playing the role of 'time'. By the strong Markov property, the BM $\widetilde{B}(t) = B(\tau_x + t) - B(\tau_x), t \geq 0$, is independent of the history $\mathcal{F}_{\tau_x}$ before hitting level x. For $y > 0$, the first passage of $\widetilde{B}$ through level y is the first passage of B through level $x + y$. It follows that τ_{x+y} can be represented as $\tau_{x+y} = \tau_x + \tau'_y$, where τ_x and τ'_y are independent, and τ'_y has the same distribution as τ_y.

This argument shows that the first passage process $(\tau_x, x \geq 0)$ has the property of independence of increments, like the BM. However, despite this similarity, the process is very different from the BM. Firstly, τ_x is nondecreasing in x, while the BM fluctuates. Secondly, $(\tau_x, x \geq 0)$ has discontinuous paths[7].

6.3 Distribution of the first passage time

Perhaps, the most striking feature of the first passage time τ_x is that $\mathbb{E}\tau_x = \infty$. The BM (starting at 0) needs, on the average, infinite time to hit any fixed level, no matter how small $|x|$.

Throughout we shall consider $x > 0$. One approach to the distribution of τ_x exploits the gBM $Z(t) = \exp(\sigma B(t) - \sigma^2 t/2)$ with $\sigma > 0$. By Theorem 6.3 $Z(\tau_x \wedge t)$ is a martingale, and

$$1 = Z(0) = \mathbb{E}Z(\tau_x \wedge t) = \mathbb{E}\left[\exp\left(\sigma B(\tau_x \wedge t) - \frac{1}{2}\sigma^2(\tau_x \wedge t)\right)\right].$$

We've seen already that $\tau_x < \infty$ a.s., but let us derive this anew. On the event $\{\tau < \infty\}$ we have $B(\tau_x \wedge t) = x$ for $t > \tau_x$, thus

$$\exp\left(\sigma B(\tau_x \wedge t) - \frac{1}{2}\sigma^2(\tau_x \wedge t)\right) \to \exp\left(\sigma x - \frac{1}{2}\sigma^2 \tau_x\right), \quad \text{as } t \to \infty.$$

On the event $\{\tau_x = \infty\}$ we have $B(\tau_x \wedge t) < x$, thus

$$\exp\left(\sigma B(\tau_x \wedge t) - \frac{1}{2}\sigma^2(\tau_x \wedge t)\right) \leq \exp\left(\sigma x - \frac{1}{2}\sigma^2 t\right) \to 0, \quad \text{as } t \to \infty.$$

Both cases can be captured by writing

$$\exp\left(\sigma B(\tau_x \wedge t) - \frac{1}{2}\sigma^2(\tau_x \wedge t)\right) \to 1(\tau_x < \infty)\exp\left(\sigma x - \frac{1}{2}\sigma^2 \tau_x\right), \quad \text{as } t \to \infty.$$

The LHS is bounded by $e^{\sigma x}$, which by the virtue of the dominated convergence theorem justifies applying $\mathbb{E}$ on both sides to obtain

$$1 = \mathbb{E}\left[1(\tau_x < \infty)\exp\left(\sigma x - \frac{1}{2}\sigma^2 \tau_x\right)\right]. \tag{36}$$

Using dominated convergence once again (say with the bound e^{2x} for $\sigma < 2$), we let $\sigma \to 0$ to obtain

$$1 = \mathbb{E}[1(\tau_x < \infty)] = \mathbb{P}(\tau_x < \infty),$$

[7] If x occurs to be a local maximum of the BM, then τ_x has a jump at x, with the jump-size equal to the time elapsed after τ_x needed for the BM to return to the level x.

which means that τ_x is finite.

Re-writing (36) as

$$\mathbb{E}\left[\exp\left(-\frac{1}{2}\sigma^2\tau_x\right)\right] = e^{-\sigma x},$$

and introducing variable $\lambda = \sigma^2/2$ we obtain the Laplace transform[8] of τ_x

$$\mathbb{E}e^{-\lambda\tau_x} = e^{-x\sqrt{2\lambda}},\ \lambda > 0, \tag{37}$$

for $x > 0$. By symmetry of the BM for arbitrary x

$$\mathbb{E}e^{-\lambda\tau_x} = e^{-|x|\sqrt{2\lambda}}, \quad x \in \mathbb{R}, \lambda > 0.$$

Differentiating the Laplace transform at 0 we obtain $\mathbb{E}\tau_x = \infty$: the time for BM to hit level $x \neq 0$ has infinite mean. The distribution of τ_x can be obtained by inverting the Laplace transform. We prefer, however, a more instructive way based on the reflection principle.

Consider a path of BM on $[0, t]$, that crosses level x (in which case $\tau_x \leq t$) and ends below y at time t (that is $B(t) \leq y$) for some given $y \leq x$. Let us reflect the part of this path on $[\tau_x, t]$ about the horizontal line at level x. This yields another path which terminates at time t above the level $2x - y$ (the new path crosses level x, because $B(0) = 0 < x \leq 2x - y$). The *reflection principle* says that this operation preserves the probability:

$$\mathbb{P}(\tau_x \leq t, B(t) \leq y) = \mathbb{P}(B(t) \geq 2x - y), \quad x > 0, y \leq x. \tag{38}$$

Theorem 6.10. *The density of τ_x is*

$$f_{\tau_x}(t) = \frac{|x|}{t^{3/2}\sqrt{2\pi}} e^{-x^2/(2t)}. \tag{39}$$

Proof. Fix $x > 0$, and use (38) with $x = y$

$$\mathbb{P}(\tau_x \leq t, B(t) \leq x) = \mathbb{P}(B(t) \geq x) = \mathbb{P}(B(t) \geq x, \tau \leq t),$$

where the last equality holds since $B(t) \geq x$ implies $\tau_x \leq t$. Changing the sides in the second equality and adding with the first yields

$$\mathbb{P}(\tau_x \leq t) = 2\mathbb{P}(B(t) \geq x) = \frac{2}{\sqrt{2\pi t}} \int_x^\infty e^{-y^2/(2t)} dy = \frac{2}{\sqrt{2\pi}} \int_{x/\sqrt{t}}^\infty e^{-z^2/2} dz, \tag{40}$$

the last by the change of variable $z = y/\sqrt{t}$. It remains to differentiate in t to obtain the density. $\square$

It should be stressed that x in (39) is a parameter. The distribution with density (39) is called *stable* [9].

[8] Another name for the Laplace transform of a random variable is the ‘moment generating function.

[9] Also known as 1/2-*stable*. The square root function appears as $\sqrt{\lambda}$ in the Laplace transform (37).

6.4 Running maximum of the BM

Let $M(t) = \max_{s\in[0,t]} B(s)$ be the running maximum of BM, and τ_x the first passage over level $x > 0$. Since $\tau_x \leq t$ holds exactly when $M(t) \geq x$, from (40) follows

$$\mathbb{P}(M(t) \geq x) = 2\mathbb{P}(B(t) \geq x) = \frac{2}{\sqrt{2\pi t}} \int_x^\infty e^{-y^2/(2t)} dy.$$

Differentiating in t and taking with minus sign, the density of $M(t)$ emerges

$$f_{M(t)}(x) = \frac{2}{\sqrt{2\pi t}} e^{-y^2/(2t)}.$$

By symmetry of the normal distribution this is also the density of $|B(t)|$, so remarkably $M(t) \stackrel{d}{=} |B(t)|$.

The absolute value process $(|B(t)|, t \geq 0)$ is sometimes called the *reflected* BM. The process is Markov (exercise). A deeper connection of the BM, its running maximum and the absolute value shows the following theorem due to Lévy.

Theorem 6.11. *Let $X(t) := M(t) - B(t)$. The process $(X(t), t \geq 0)$ has the same distribution as the process $(|B(t)|, t \geq 0)$.*

The idea of the proof

The maximum process $(M(t), t \geq 0)$ itself is not a Markov process. However, the inverse function, which we may write as

$$\tau_x = \min\{t : M(t) \geq x\}, \quad x \geq 0,$$

is the Markov process (with x as 'time' parameter). Note that each time interval of length, say ℓ, where the running maximum is constant, corresponds to a jump of the inverse process: of $\lim_{y\downarrow x}(\tau_y - \tau_x) = \ell$ (where $y \downarrow x$ means 'as y decreases to x').

Theorem 6.12. *The joint density of $M(t), B(t)$ is*

$$f_{M(t),B(t)}(x, y) = \frac{2(2x - y)}{t\sqrt{2\pi t}} \exp\left(-\frac{(2x - y)^2}{2t}\right), \quad x \geq y, \ x \geq 0. \tag{41}$$

Proof.

$$\mathbb{P}(M(t) \geq x, B(t) \leq y) = \mathbb{P}(B(t) \geq 2x - y) = \frac{1}{\sqrt{2\pi t}} \int_{2x-y}^\infty e^{-z^2/(2t)} dz.$$

Calculating first the partial derivative in x

$$-\frac{\partial}{\partial x} \mathbb{P}(M(t) \geq x, B(t) \leq y) = \frac{-2}{\sqrt{2\pi t}} \exp\left(-\frac{(2x - y)^2}{2t}\right),$$

then differentiating in y gives the formula (41). □

6.5 Maximum of the BM with drift

We turn next to the BM with drift $\widetilde{B}(t) = \alpha t + B(t)$ and its running maximum $\widetilde{M}(t) = \max_{s\in[0,t]} \widetilde{B}(t)$. Clearly, $\widetilde{M}(t) \geq \widetilde{B}(0) = 0$, therefore the vector $(\widetilde{M}(t), \widetilde{B}(t))$ assumes values in the set of vectors (x, y) such that $x \geq 0,\ y \leq x$.

The following result extends Theorem 41

Theorem 6.13. *The joint density of* $(\widetilde{M}(t), \widetilde{B}(t))$ *is*

$$f_{\widetilde{M}(t),\widetilde{B}(t)}(x,y) = \frac{2(2x-y)}{t\sqrt{2\pi t}} \exp\left(\alpha y - \frac{1}{2}\alpha^2 t - \frac{1}{2t}(2x-y)^2\right). \tag{42}$$

Proof. The clue is Girsanov's Theorem 4.3. Introduce $Z(t) = \exp(-\alpha B(t) - \alpha^2 t/2)$ and note that in terms of the drifted BM $Z(t) = \exp(-\alpha\widetilde{B}(t)+\alpha^2 t/2)$ Changing the probability measure to $\widetilde{\mathbb{P}}$ with the Radon-Nikodym derivative Z, we achieve that $\widetilde{B}$ is the standard BM under $\widetilde{\mathbb{P}}$. The expectations under the measures are connected as $\mathbb{E}\xi = \widetilde{\mathbb{E}}[\xi/Z]$, which implies

$$\mathbb{P}(\widetilde{M}(t) \leq x, \widetilde{B}(t) \leq y) = \mathbb{E}[1(\widetilde{M}(t) \leq x, \widetilde{B}(t) \leq y)] =$$
$$\widetilde{\mathbb{E}}\left[\frac{1(\widetilde{M}(t) \leq x, \widetilde{B}(t) \leq x)}{Z(t)}\right] = \widetilde{\mathbb{E}}\left[\frac{1(\widetilde{M}(t) \leq x, \widetilde{B}(t) \leq x)}{\exp(-\alpha\widetilde{B}(t) + \alpha^2 t/2)}\right] =$$
$$\int_{-\infty}^{y}\int_{-\infty}^{x} \exp(\alpha v - \alpha^2 t/2) f_{M(t),B(t)}(u,v)\,du\,dv,$$

where $f_{M(t),B(t)}$ from (41) appears as the joint density of $\widetilde{M}(t), \widetilde{B}(t)$ under $\widetilde{\mathbb{P}}$. Differentiating in x and y, the desired density of $(\widetilde{M}(t), \widetilde{B}(t))$ under $\mathbb{P}$ is $\exp(\alpha y - \alpha^2 t/2) f_{M(t),B(t)}(x,y)$. $\square$

Tedious but straightforward calculation (see Shreve's book pp. 297-299) allows to evaluate the integral in terms of the normal distribution function Φ:

$$\mathbb{P}(\widetilde{M}(t) \leq x) = \int_0^x \int_{-\infty}^x f_{\widetilde{M}(t),\widetilde{B}(t)}(u,v)\,du\,dv = \Phi\left(\frac{x-\alpha t}{\sqrt{t}}\right) - e^{-2\alpha x}\Phi\left(\frac{-x-\alpha t}{\sqrt{t}}\right).$$

Differentiating in x gives the density of $\widetilde{M}(t)$

$$f_{\widetilde{M}(t)}(x) = \frac{2}{\sqrt{2\pi t}} e^{-(x-\alpha t)^2/(2t)} - 2\alpha e^{-2\alpha x}\Phi\left(\frac{-x-\alpha t}{\sqrt{t}}\right).$$

On the other hand, we may consider the first passage time of the drifted BM $\widetilde{B}$ over level $x > 0$

$$\widetilde{\tau}_x = \min\{t : \widetilde{B}(t) = x\},$$

for which $\{\widetilde{\tau}_x \geq t\} = \{\widetilde{M}(t) \leq x\}$, so

$$\mathbb{P}(\widetilde{\tau}_x \geq t) = \Phi\left(\frac{x-\alpha t}{\sqrt{t}}\right) - e^{-2\alpha x}\Phi\left(\frac{-x-\alpha t}{\sqrt{t}}\right).$$

Suppose first $\alpha < 0$, the drift is negative. Then $\Phi\left(\frac{\pm x - \alpha t}{\sqrt{t}}\right) \to 1$ as $t \to \infty$, and

$$\mathbb{P}(\widetilde{\tau}_x \geq t) \to \mathbb{P}\widetilde{\tau}_x = \infty) = 1 - e^{2\alpha x},$$

which is the probability that $\widetilde{B}$ never passes x. That this probability is positive should not be surprising: $\widetilde{B}(t)$ drifts down to $-\infty$ as $t \to \infty$, hence does not reach sufficiently high levels. In this case the *total maximum* $\max_{t\in[0,\infty)} \widetilde{B}(t)$ is a random variable with the exponential distribution of rate 2α. That the distribution of total maximum must be exponenial, could be guessed from the strong Markov property of $\widetilde{B}$ combined with the memorylessness of the exponential distributions.

In the case $\alpha > 0$, $\Phi\left(\frac{\pm x - \alpha t}{\sqrt{t}}\right) \to 0$ as $t \to \infty$, and $\widetilde{\tau}_x$ is finite a.s. Differentiating we obtain the density

$$f_{\widetilde{\tau}_x}(t) = \frac{x}{\sqrt{2\pi t^3}} \exp\left(-\frac{(x - at)^2}{2t}\right).$$

The distribution with this density is called *inverse Gaussian*. For $\alpha = 0$ we are back to the 1/2-stable distribution (39) of the BM passage time τ_x.

6.6 Pricing a barrier option

Up-and-out call option on the stock is an option which pays to the holder at maturity $(S(T) - K)^+$ provided the stock price was never above the barrier level L. If $S(t) > L$ for some $t \in [0, T]$ the option is worthless. The option is of European type. We assume $L > K$, as otherwise the option pays zero in any case. Compared with the standard call, the barrier option sets a bound bounds on the liability of the option seller.

Let $\widetilde{\mathbb{P}}$ we the risk-neutral probability measure, and consider the stock driven under this measure by the equation

$$dS(t) = rS(t)dt + \sigma S(t)dB(t),$$

where as usual r is the riskless bank rate, σ is the volatility.

To spare notation, we shall consider the stock with initial price $S(0) = 1$. This is no loss of generality, as we can always rescale the strike and the barrier to the values $K/S(0)$ and $L/S(0)$. The stock price is the gBM

$$S(t) = e^{\sigma \widetilde{B}(t)},$$

with $\widetilde{B}(t) := (r/\sigma - \sigma/2)t + B(t)$. Introducing $k = \sigma^{-1} \log K, b = \sigma^{-1} \log L$, the option pays at the maturity

$$V(T) = \left(e^{\sigma \widetilde{B}(T)} - K\right) 1(\widetilde{B}(T) \geq k, \widetilde{M}(T) \leq b),$$

where $\widetilde{M}(t) := \max_{x\in[0,T]} B(t)$. The knock out condition amounts to $\widetilde{M}(T) > b$.

The risk-neutral pricing of the option dictates that the price at time $t \leq T$ should be

$$V(t) = \widetilde{\mathbb{E}}[e^{-r(T-t)}V(T)|\mathcal{F}_t],$$

so that $e^{-rt}V(t)$ is a martingale. Naturally, $V(t)$ depends on $S(t)$, but not only. The terminal payoff $V(T)$ depends on $S(T)$ and the largest stock price before expiration $\exp(\sigma\widetilde{M}(T))$. The pair $(S(t),\widetilde{M}(t))$ is a bivariate Markov process, hence the conditional discounted payoff $V(t)$ can be represented as a function of $(S(t),\widetilde{M}(t))$. Furthermore, if $\widetilde{M}(t) > b$ (then the running maximum of the stock price is larger L) the option is worthless and $V(t)=0$, while if $\widetilde{M}(t)\le b$ the value of the option is $V(t)=v(t,S(t))$ for some function v, because in the latter case $\widetilde{M}(T)<b$ is the same as $\max_{u\in[t,T]}S(u)<L$. The two cases are captured by the formula $V(t)=v(t,S(t))\,1(\widetilde{M}(t)\le b)$.

More explicitly, the function $v(t,x)$ is the conditional expectation

$$v(t,x)=\widetilde{\mathbb{E}}\left[e^{-r(T-t)}1(\max_{u\in[t,T]}S(u)\le L)(S(T)-K)^+|S(t)=x\right],\quad x\in[0,L].$$

The martingale property of $V(t), t\in[0,T]$ implies that the stochastic differential of $V(t)=v(t,S(t))\,1(\widetilde{M}(t)\le b)$ should have no 'dt' term, which allows to conclude[10] that in the domain $(t,x)\in[0,T]\times[0,L]$ the function v satisfies the Black-Scholes PDE

$$v_t(t,x)+rxv_x(t,x)+\frac{1}{2}\sigma^2x^2v_{xx}(t,x)=rv(t,x),$$

which should be complemented by the obvious boundary conditions

$$\begin{aligned}v(t,0)&=0,\quad 0\le t\le T,\\ v(t,B)&=0,\quad 0\le t\le T,\\ v(T,x)&=(x-K)^+,\quad 0\le x\le B.\end{aligned}$$

To evaluate $V(0)=\widetilde{\mathbb{E}}[e^{-rT}V(T)]$ explicitly we need to integrate the discounted payoff weighted with the joint density (42) of $(\widetilde{M}(T),\widetilde{B}(T))$

$$V(0)=e^{-rT}\int_k^b\int_{y^+}^b(e^{\sigma y}-K)\frac{2(2x-y)}{T\sqrt{2\pi T}}\exp\left(\alpha y-\frac{1}{2}\alpha^2T-\frac{1}{2T}(2x-y)^2\right)dydx$$

The domain of integration is $\{(x,y): y^+\le x\le b,\ k\le y\le b\}$, where as usual $y^+=\max(y,0)$. The assumption $S(0)=1\le L$ means $b>0$, but $k<0$ is not excluded (the original stock price $S(0)<K$).

Introducing

$$\delta_\pm(y):=\frac{1}{\sigma\sqrt{T}}(\log y+(r\pm\frac{1}{2}\sigma^2)T)$$

the value is (for the chosen case $S(0)=1$)

$$\begin{aligned}V(0)=[\Phi(\delta_+(K^{-1}))-\Phi(\delta_+(B^{-1})]-e^{-rT}K[\Phi(\delta_-(K^{-1}))-\Phi(\delta_-(L^{-1}))]-\\ L^{2r/\sigma^2+1}[\Phi(\delta_+(L^2K^{-1}))-\Phi(\delta_-(L))]+e^{-rT}KL^{2r/\sigma^2-1}[\Phi(\delta_-(L^2K^{-1}))-\Phi(\delta_-(L))].\end{aligned}$$

We refer to Shreve's textbook (Section 7.3.3) for details of calculation of the integral. Scaling properly the variables it is not hard to derive a similar formula for $v(t,x)$.

[10] A rigorous argument employs the optional sampling theorem.

6.7 American call option

Option of the American type can be exercised any time t before it expires at given time T. The investor holding the option faces a problem of choosing the optimal time to exercise the option. This exercise time is a stopping time in the sense of Definition 6.7. In general, the American option is more valuable than the analogous European option, since the strategy of exercising at time T pays the same as the European counterpart.

The American call with strike K pays when the option is exercised the amount $(S(t)-K)^+$. The problem of finding the optimal stopping time in this case turns very simple: it is optimal to exercise the option at the maturity time T. Thus the American call brings no advantage over the European call with the same K, T. A clue to pricing the American call is *convexity*.

More generally, let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a nonnegative convex function with $h(0) = 0$ [11].

The convexity means that

$$h(px + qy) \leq ph(x) + qh(y)$$

for arbitrary nonnegative p, q with $p + q = 1$. In particular, choosing $y = 0$ we get $h(px) \leq ph(x)$ for $0 \leq p \leq 1$.

Consider the option which pays $h(S(t))$ if exercised at time $t \leq T$. Assume the stock under the risk-neutral measure $\widetilde{\mathbb{P}}$ is driven by a gBM with drift r and volatility σ. Recall that the discounted stock price $e^{-rt}S(t)$ is a martingale under $\widetilde{\mathbb{P}}$.

A *submartingale* is a random process X which tends to increase, that is satisfies $\mathbb{E}[X(t)|\mathcal{F}_u] \geq X(u)$ for $u < t$. The analogue of Theorem 6.8 holds for submartingales in the form $\mathbb{E}[X(\tau_2)|\mathcal{F}_{\tau_1}] \geq X(\tau_2)$ for two finite stopping times $\tau_1 \leq \tau_2$.

Lemma 6.14. *The discounted intrinsic value process* $(e^{-rt}h(S(t)),\ t \in [0, T])$ *is a submartingale:*

$$\widetilde{\mathbb{E}}(e^{-rt}h(S(t))|\mathcal{F}_u) \geq e^{-ur}h(S(u)), \quad 0 \leq u \leq t \leq T.$$

Proof. For $t > u$ setting $p = e^{-r(t-u)}$ in the above implies

$$\widetilde{\mathbb{E}}[e^{-r(t-u)}h(S(t))|\mathcal{F}_u] \geq \widetilde{\mathbb{E}}[h(e^{-r(t-u)}S(t))|\mathcal{F}_u].$$

By convexity of h, Jensen's inequality applies

$$\widetilde{\mathbb{E}}[h(e^{-r(t-u)}S(t))|\mathcal{F}_u] \geq h\left(\widetilde{\mathbb{E}}[(e^{-r(t-u)}S(t))|\mathcal{F}_u]\right) =$$
$$h\left(e^{ru}\widetilde{\mathbb{E}}[(e^{-rt}S(t))|\mathcal{F}_u]\right) = h(e^{ru}e^{-ru}S(u))h(S(u)),$$

where we used that $e^{-rt}S(t), t \geq 0$, is a martingale.

□

Theorem 6.15. *Suppose an option of American type pays at the maturity* $h(S(T))$, *where* h *is a nonnegative convex function with* $h(0) = 0$. *Then the risk-neutral price of the option is the same as of the analogous European option.*

[11] The condition $h(0) = 0$ is natural, but not restrictive, since can always be achieved by switching to $\hat{h}(x) = h(x) - h(0)$.

Proof. For every $t < T$

$$\widetilde{\mathbb{E}}(e^{-rT}h(S(T))|\mathcal{F}_t) \geq e^{-rt}h(S(t)), \quad 0 \leq u \leq t \leq T.$$

The inequality still holds if the fixed time t is replaced by a random stopping time τ with values in $[0, T]$. $\square$

The American call option appears as the special case of the result due to convexity of the functions $h(x) = (x - K)^+$.

The problem of pricing the American put option, with $h(x) = (K - x)^+$ is much more complicated and has no closed-form solution. The put-call parity, useful for the European options, does not apply to the American options, because always holding the American put till expiration of the option is not the optimal strategy.