4 Risk-neutral pricing

We start by discussing the idea of risk-neutral pricing in the framework of the elementary one-step binomial model. Suppose there are two times $t = 0$ and $t = 1$. At time 0 the stock has value $S(0)$ and at time 1 either goes up to $S(1) = uS(0)$ or down to $S(1) = dS(0)$, where $0 < d < 1 + r < u$ and r the riskless return rate in the money market. A pound invested at time 0 grows to $1 + r$ at time 1. Suppose an option pays $V(S(1))$ at time 1.

To hedge the option we need a portfolio comprised of

$$\mathit{\Delta}(0) = \frac{V(uS(0)) - V(dS(0))}{uS(0) - dS(0)}$$

shares of stock and a bank investment, so that the total value of the portfolio at time 0 is

$$X(0) = \frac{1}{1+r}[\tilde{p}V(uS(0)) + \tilde{q}V(dS(0))],$$

where $\tilde{p} = (1 + r - d)/(u - d), \tilde{q} = 1 - \tilde{p}$. This is readily checked by elementary algebra.

For the purpose of hedging, leading to the no-arbitrage valuation of the option, it does not matter what is the 'market measure' = the chances that stock moves up/down [2]. We may, however, introduce an auxiliary *risk-neutral measure* $\widetilde{\mathbb{P}}$ with chances $\tilde{p}$ respectively $\tilde{q}$ for 'up' and 'down', and write the no-arbitrage price of the option as discounted expected value

$$X(0) = \frac{1}{1+r}\widetilde{\mathbb{E}}V,$$

where and henceforth the expectation $\widetilde{\mathbb{E}}$ refers to the probability $\widetilde{\mathbb{P}}$. If the stock process *were* driven by the RN measure, investment in the stock would bring the same mean rate of return r as the money market.

If $\mathbb{P}$ is a 'market measure' under which the stock goes up and down with probabilities p, respectively $q = 1 - p$, then in terms of $\mathbb{P}$ we could also write

$$X(0) = \frac{1}{1+r}\mathbb{E}(ZV),$$

where Z is a random variable equal to $\tilde{p}/p$ if the stock goes up and equal to $\tilde{q}/q$ is the stock goes down. This manipulation of $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ is called the *change of measure*. Note that rv Z needed to pass from $\mathbb{P}$ to $\widetilde{\mathbb{P}}$ is well defined when $p > 0, q > 0$.

Under the measure $\widetilde{\mathbb{P}}$ the discounted stock price is a martingale, which amounts to the simple relation

$$\widetilde{\mathbb{E}}\left(\frac{1}{r+1}S(1)\right) = S(0),$$

and means that in the $\widetilde{\mathbb{E}}$-average the stock behaves like the riskless bond growing from $S(0)$ to $(1 + r)S(0)$ in unit time.

In continuous time model, suppose that the interest on money investment is continuously compounded at rate r. The idea of risk-neutral pricing in continuous time is, in

[2] It is only important that going both ways occurs with nonzero probability.

principle, the same as in the elementary binomial model: we look for a probability measure $\widetilde{\mathbb{P}}$ such that the discounted stock price $e^{-rt}S(t)$ under $\widetilde{\mathbb{P}}$ is a martingale. Consider a European type derivative security which pays $V(T)$ at time T, and let $X(t)$ be the value of hedging portfolio, so $X(T) = V(T)$. For any self-financing portfolio its discounted value is a $\widetilde{\mathbb{P}}$-martingale, because the mean rate of return is the same for both money and stock, thus

$$e^{-rt}X(t) = \widetilde{\mathbb{E}}\left(e^{-rT}X(T)|\mathcal{F}_t\right) = \widetilde{\mathbb{E}}\left(e^{-rT}V(T)|\mathcal{F}_t\right).$$

Practically that means that the no-arbitrage option price is computable as a conditional expectation of the terminal payoff.

4.1 Girsanov's theorem

We have seen in our discussion of the BSM formula that the price of a European call is an expected value calculated for some gBM, but not the original gBM describing the stock price. The expected value refers to the *risk-neutral measure*, which is a probability law of the stock price process, under which (on the average) the investor cannot outperform the riskless money market (bonds). The relation between two probability laws can be described using the change of measure technique.

For given probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a nonnegative random variable Z with $\mathbb{E}Z = 1$ a new probability measure $\widetilde{\mathbb{P}}$ is defined by

$$\widetilde{\mathbb{P}}(A) = \mathbb{E}(Z \cdot 1_A), \qquad A \in \mathcal{F},$$

where 1_A is the indicator of the event A. We say that Z is a Radon-Nikodym density (or derivative) of $\widetilde{\mathbb{P}}$ with respect to $\mathbb{P}$ and we write

$$Z = \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}.$$

If $\mathbb{P}(Z > 0) = 1$ then the measures are *equivalent*: $\mathbb{P}(A) = 0$ iff $\widetilde{\mathbb{P}}(A) = 0$ for $A \in \mathcal{F}$. In this case passing in the inverse direction, from $\widetilde{\mathbb{P}}$ to $\mathbb{P}$ requires the Radon-Nikodym derivative $Z^{-1} = d\mathbb{P}/d\widetilde{\mathbb{P}}$, so that $\mathbb{P}(A) = \widetilde{\mathbb{E}}(Z^{-1} \cdot 1_A)$.

If $\mathbb{P}(\omega) > 0$ for elementary event $\omega \in \Omega$, then $\widetilde{\mathbb{P}}(\omega) = Z(\omega)\mathbb{P}(\omega)$. This describes completely the relation between the measures if Ω is a discrete sampling space. We encountered this situation in the above 1-step binomial model example, where we could take a two-element sampling space $\Omega = \{\text{up}, \text{down}\}$ and define the random variable $S(1)$ as $S(1, \text{up}) = uS(0), S(1, \text{down}) = dS(0)$. But if Ω is not discrete the relation between the measures on Ω cannot be expressed through probabilities of the elementary events, as typically $\mathbb{P}(\omega) = 0$ for each $\omega \in \Omega$. The Radon-Nikodym derivative in the continuous case is analogous to the concept of density of a random variable in one or many dimensions.

The expected values of a random variable ξ under $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are connected as

$$\widetilde{\mathbb{E}}\xi = \mathbb{E}(\xi Z),$$

and if $\mathbb{P}(Z > 0) = 1$ also as $\mathbb{E}\xi = \widetilde{\mathbb{E}}(\xi Z^{-1})$. In particular, $\widetilde{\mathbb{E}}1 = \mathbb{E}Z = 1$, i.e. $\widetilde{\mathbb{P}}$ is indeed a probability measure (the total probability rule holds: $\widetilde{\mathbb{P}}(\Omega) = 1$).

Under $\mathbb{P}$ and $\widetilde{\mathbb{P}}$, the same random variable has typically different distributions.

Example 4.1. Suppose under a random variable ξ, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ has standard normal distribution: $\xi \stackrel{d}{=} \mathcal{N}(0,1)$. This means that

$$\mathbb{E}f(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-x^2/2}dx$$

for all functions f (for which the integral makes sense). Let $Z = \exp(-\theta\xi - \frac{1}{2}\theta^2) = d\widetilde{\mathbb{P}}/d\mathbb{P}$ be a Radon-Nikodym derivative. We have

$$\widetilde{\mathbb{E}}f(\xi) = \mathbb{E}(f(\xi)Z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-x^2/2}e^{-\theta x - \frac{1}{2}\theta^2}dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-(x+\theta)^2/2}dx,$$

where the last integral involves the $\mathcal{N}(-\theta, 0)$-density. Thus under $\widetilde{\mathbb{P}}$ the distribution of ξ is $\mathcal{N}(-\theta, 0)$. Equivalently, $\xi + \theta$ under $\widetilde{\mathbb{P}}$ is standard normal. To justify that Z is indeed a Radon-Nikodym derivative we note that $z \geq 0$ and $\widetilde{\mathbb{E}}\xi = 1$ (take $f = 1$ to see this).

The example showed that via changing measure we could shift the expectation of a normal variable. We wish to do a similar change of measure to shift the expectation for a whole random process of Brownian motion with drift.

Let Z be a random variable with $\mathbb{P}(Z > 0) = 1$ and $\mathbb{E}Z = 1$, and let $(\mathcal{F}_t, t \in [0,T])$ be a filtration of σ-algebras. Define the conditional expectation

$$Z(t) = \mathbb{E}[Z|\mathcal{F}_t], \quad 0 \leq t \leq T,$$

which considered as a random process is called the Radon-Nikodym derivative process. By the 'tower property' of conditional expectations, for $0 \leq s \leq t \leq T$

$$\mathbb{E}[Z(t)|\mathcal{F}_s] = \mathbb{E}[\mathbb{E}[Z|\mathcal{F}_t]\mathcal{F}_s] = \mathbb{E}[Z|\mathcal{F}_s] = Z(s),$$

so $(Z(t), t \in [0,T])$ is a martingale.

Lemma 4.2. *For $\mathcal{F}_t$-measurable random variable Y it holds*

(i) $\widetilde{\mathbb{E}}Y = \mathbb{E}(YZ(t))$,

(ii) $\widetilde{\mathbb{E}}[Y|\mathcal{F}_s] = \frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}_s], \quad 0 \leq s \leq t \leq T.$

Proof. (i) By the definition of $\widetilde{\mathbb{E}}$ and since Y is $\mathcal{F}_t$-measurable

$$\widetilde{\mathbb{E}}Y = \mathbb{E}[YZ] = \mathbb{E}[\mathbb{E}[YZ|\mathcal{F}_t]] = \mathbb{E}[Y\mathbb{E}[Z|\mathcal{F}_t]] = \mathbb{E}[YZ(t)].$$

(ii) By definition of the conditional expectation, $\widetilde{\mathbb{E}}[Y|\mathcal{F}_s]$ is a $\mathcal{F}_s$-measurable random variable with the property that

$$\widetilde{\mathbb{E}}(1_A \cdot \widetilde{\mathbb{E}}[Y|\mathcal{F}_s]) = \widetilde{\mathbb{E}}[1_A Y], \quad \text{for all } A \in \mathcal{F}_s.$$

Using the relation $\widetilde{\mathbb{E}}\xi = \mathbb{E}(Z\xi)$, that $Z(s), 1_A$ are $\mathcal{F}_s$-measurable and the definition of $\mathbb{E}(\cdots|\mathcal{F}_s)$

$$\widetilde{\mathbb{E}}\left(1_A \frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}_s]\right) = \mathbb{E}\left(1_A \mathbb{E}[YZ(t)|\mathcal{F}_s]\right) =$$
$$\mathbb{E}\left(\mathbb{E}[1_A YZ(t)|\mathcal{F}_s]\right) = \mathbb{E}[1_A YZ(t)] = \widetilde{\mathbb{E}}[1_A Y],$$

where the last step follows from (i), and because the indicator 1_A is $\mathcal{F}_s$- hence also $\mathcal{F}_t$-measurable. $\square$

In the sequel we shall be using the Leibnitz rule for stochastic differentials of the product of two Ito processes

$$d(X(t)Y(t)) = Y(t)dX(t) + X(t)dY(t) + dX(t)dY(t).$$

This can be derived (somewhat heuristically) by expanding $(X(t)+dX(t))(Y(t)+dY(t))-X(t)Y(t)$ and using that both $(dX(t))^2$ and $(dY(t))^2$ are of the order of dt.

The change of measure technique allows us to transform a BM with drift in the standard BM.

Theorem 4.3. (Girsanov's Theorem) *Suppose B defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a BM with filtration $(\mathcal{F}_t, t \geq 0)$, and let $(\mu(t), t \geq 0)$ be an adapted process. Define*

$$\widetilde{B}(t) := B(t) + \int_0^t \mu(u)du$$

and

$$Z(t) := \exp\left(-\int_0^t \mu(u)dB(u) - \frac{1}{2}\int_0^t \mu^2(u)du\right), \quad Z := Z(T). \tag{21}$$

Suppose that

$$\mathbb{E}\int_0^T \mu^2(t)Z^2(t)dt < \infty.$$

Then under the probability measure $\widetilde{\mathbb{P}}$ with the Radon-Nikodym density $d\widetilde{\mathbb{P}}/d\mathbb{P} = Z$ the process $(\widetilde{B}(t), t \in [0,T])$ is a standard BM.

Proof. We shall employ a characterisation theorem by Lévy, which asserts that a continuos martingale starting at 0 and having the quadratic variation equal to t for every $t > 0$ is a BM. It is clear that $\widetilde{B}(0) = 0$ and that the quadratic variation is $\langle\widetilde{B}\rangle(t) = t$, because the quadratic variation is a property of the path (and not of the distribution of the process) and because it is not affected by the drift term.

To apply Lévy's theorem it remains to show that $\widetilde{B}$ is a martingale under $\widetilde{\mathbb{P}}$.

By definition, $Z(t)$ is a generalised gBM $Z(t) = e^{X(t)}$ where

$$X(t) = -\int_0^t \mu(u)dB(u) - \frac{1}{2}\int_0^t \mu^2(u)du.$$

By the Ito formula

$$dZ(t) = de^{X(t)} = e^{X(t)}(-\mu(t)dB(t) - \frac{1}{2}\mu^2(t)dt) + \frac{1}{2}e^{X(t)}\mu^2 dt = -\mu(t)Z(t)dB(t),$$

which means that $Z(t)$ is a stochastic integral

$$Z(t) = Z(0) - \int_0^t \mu(u)Z(u)dB(u),$$

hence martingale under $\widetilde{\mathbb{P}}$. In particular $\mathbb{E}Z = \mathbb{E}Z(T) = Z(0) = 1$. The assumption of square-integrability has been used to justify that the stochastic integral is well defined.

By the martingale property,

$$Z(t) = \mathbb{E}[Z(T)|\mathcal{F}_t] = \mathbb{E}[Z|\mathcal{F}_t]$$

is a Radon-Nikodym derivative process.

Applying the product rule $d(XY) = XdY + YdX + dXdY$ we obtain

$$\begin{aligned} d(\widetilde{B}(t)Z(t)) = \widetilde{B}(t)dZ(t) + Z(t)d\widetilde{B}(t) + d\widetilde{B}(t)dZ(t) = \\ -\widetilde{B}(t)\mu(t)Z(t)dB(t) + Z(t)dB(t) + Z(t)\mu(t)dt + (dB(t) + \mu(t)dt)(-\mu(t)Z(t)dB(t)) = \\ (-\widetilde{B}(t)\mu(t) + 1)Z(t)dB(t). \end{aligned}$$

There is no drift dt-term, hence $\widetilde{B}(t)Z(t)$ is a martingale under $\mathbb{P}$.

Finally, using the martingale property of $\widetilde{B}(t)Z(t)$ under $\mathbb{P}$ and using Lemma 4.2 , for $0 \leq s \leq t \leq T$ we have

$$\widetilde{\mathbb{E}}[\widetilde{B}(t)|\mathcal{F}_s] = \frac{1}{Z(s)}\mathbb{E}[\widetilde{B}(t)Z(t)|\mathcal{F}_s] = \frac{1}{Z(s)}\widetilde{B}(s)Z(s) = \widetilde{B}(s),$$

which is the desired martingale property of $\widetilde{B}$ under $\widetilde{\mathbb{P}}$.

□

For example, suppose μ is constant. By Girsanov's theorem, for any fixed t the rv $\widetilde{B}(t) + \mu t$ under $\widetilde{\mathbb{P}}$ is $\mathcal{N}(0, t)$-distributed, so has zero expectation.

4.2 Stock price under the risk-neutral measure

A general model for the stock price is a generalised gBM with differential

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dB(t), \quad t \in [0, T], \tag{22}$$

where both $\alpha(t), \sigma(t)$ are adapted processes. In the integral form,

$$S(t) = S(0)\exp\left[\int_0^t \sigma(u)dB(u) + \int_0^t (\alpha(u) - \sigma^2(u)/2)du\right].$$

Let the interest rate be another adapted process $R(t)$. The *discount process*

$$D(t) = e^{-\int_0^t R(u)du}$$

satisfies

$$dD(t) = -R(t)D(t)dt, \tag{23}$$

which is a formula from the ordinary calculus, since we can use $d\left(\int_0^t R(u)du\right) = R(t)dt$. One pound invested at time 0 in bank becomes $1/D(t)$ at time t, or, equivalently, 1 pound at time t has present (time 0) value $D(t)$.

For the discounted stock price

$$D(t)S(t) = S(0)\exp\left(\int_0^t \sigma(u)dB(u) + \int_0^t (\alpha(u) - R(u) - \frac{1}{2}\sigma^2(u))du\right)$$

we compute

$$d(D(t)S(t)) = (\alpha(t)-R(t))D(t)S(t)dt+\sigma(t)D(t)S(t)dB(t) = \sigma(t)D(t)S(t)(\mu(t)dt+dB(t)), \quad (24)$$

where

$$\mu(t) := \frac{\alpha(t) - R(t)}{\sigma(t)}$$

is the 'market price of risk'. Calculating the differential we have set $(dD(t))(dS(t)) = 0$ since the product is of the 'order smaller than dt' in the view of (23).

With this μ, consider Z as in (21) and $\widetilde{B}(t) = B(t) + \int_0^t \mu(t)dt$. In these terms,

$$d(D(t)S(t)) = \sigma(t)D(t)S(t)d\widetilde{B}(t), \quad (25)$$

Define measure $\widetilde{\mathbb{P}}$ with this Radon-Nikodym density Z with respect to $\mathbb{P}$. By Girsanov's theorem, $\widetilde{B}$ is a BM, hence from

$$D(t)S(t) = S(0) + \int_0^t \sigma(u)D(u)S(u)d\widetilde{B}(u)$$

follows that $(D(t)S(t), t \in [0,T])$ is a martingale under $\widetilde{\mathbb{P}}$. For this reason, $\widetilde{\mathbb{P}}$ is called the *risk-neutral* measure. The mean rate of return form the stock investment under the RN measure is the same as from the bank investment.

For the undiscounted stock price under $\widetilde{\mathbb{P}}$ we have

$$dS(t) = R(t)S(t)dt + \sigma(t)S(t)d\widetilde{B}(t),$$

where the mean rate of return $R(t)dt$ is like in the money market.

4.3 Pricing under the risk-neutral measure

We continue with the general model for stock price (22) with the aim to construct hedging strategies. The value $X(t)$ of self-financing portfolio with $\Delta(t)$ shares of stock has differential

$$dX(t) = \Delta(t)dS(t)+R(t)(X(t)-\Delta(t)S(t))dt = R(t)X(t)dt+\Delta(t)\sigma(t)S(t)(\mu(t)dt+dB(t)).$$

From (23) and (24)

$$d(D(t)X(t)) = \Delta(t)\sigma(t)D(t)S(t)[\mu(t)dt + dB(t)] = \Delta(t)d(D(t)S(t)),$$

and further from (25)

$$d(D(t)X(t)) = \Delta(t)\sigma(t)D(t)S(t)d\widetilde{B}(t),$$

showing that under $\widetilde{\mathbb{P}}$ the discounted stock portfolio value $D(t)X(t)$ is a martingale, regardless of how the investor composes the portfolio (in a self-financing manner though).

In Section 3 we derived the BSM formula (20) for the price of European call option, under the assumption of constant parameters α, r, σ. We shall take now a wider approach, and consider the more general model for stock price. Another general assumption we make is that the payoff of a derivative security with maturity T is a $\mathcal{F}(T)$-measurable random variable $V(T)$. This could be any exotic option, e.g. with $V(T) = \max_{t\in[0,T]} S(t)$ (the highest stock price) or $V(T) = 1(\max_{t\in[0,T]} S(t) > b)$ (one pound if the barrier b was hit).

We postpone discussion of the *existence* of self-financing portfolio with value $(X(t), t \in [0,T])$, which hedges the option so that $X(T) = V(T)$. For a time being just assume that there exists a hedging portfolio of the kind. As $(D(t)X(t), t \in [0, Y])$ is a martingale under $\widetilde{\mathbb{P}}$ we have

$$D(t)X(t) = \widetilde{\mathbb{E}}[D(T)X(T)|\mathcal{F}_t] = \widetilde{\mathbb{E}}[D(T)V(T)|\mathcal{F}_t].$$

To avoid an arbitrage opportunity the value of option at time t should be taken as $V(t) = X(t)$. Thus we can compute $V(t)$ from the conditional RN-expectation of the terminal value

$$D(t)V(t) = \widetilde{\mathbb{E}}[D(T)V(T)|\mathcal{F}_t], \quad t \in [0,T], \tag{26}$$

that is

$$V(t) = \widetilde{\mathbb{E}}\left[e^{-\int_t^T R(u)du}V(T)|\mathcal{F}_t\right], \quad t \in [0,T].$$

BSM formula for European call Applying the RN valuation to European call, under the assumption of constant parameters α, σ, r we are lead to evaluating

$$\widetilde{\mathbb{E}}\left[e^{-r(T-t)}(S(T)-K)^+|\mathcal{F}_t\right] = S(t)\,\widetilde{\mathbb{E}}\left[e^{-r(T-t)}(S(T)/S(t) - K/S(t))^+|\mathcal{F}_t\right],$$

where under $\widetilde{\mathbb{P}}$ the gBM is representable as $S(u) = S(0)\exp(\sigma B(u) + (r - \sigma^2/2)u)$ with BM B. The conditional distribution of $S(T)/S(t)$ given $\mathcal{F}_t$ is the same as the distribution of $\exp(-\sigma\sqrt{\tau}\xi + (r-\sigma^2/2)\tau)$ where $\tau = T - t$ and $\xi \stackrel{d}{=} \mathcal{N}(0,1)$. Therefore computing the conditional expectation reduces to Exercise 7 from sheet 1, with obvious amandments in the notation.

We return to the question of existence of hedging portfolio for a derivative security paying $V(T)$. A general result on martingale processes adapted to the (natural) Brownian filtration of a BM B says that every such martingale $(M(t), t \in [0,T])$ has a stochastic integral representation

$$M(t) = M(0) + \int_0^t \lambda(u)dB(u), \quad t \in [0,T]$$

with some adapted process $\lambda(t)$.

We define $D(t)V(t)$ as the conditional expectation (26), from which this is a martingale under the RN-measure $\widetilde{\mathbb{P}}$ (check this!), so by the representation result

$$D(t)V(t) = V(0) + \int_0^t \lambda(u)d\widetilde{B}(u)$$

for some adapted process $\lambda(t)$. On the other hand, for self-financing portfolio the value $X(t)$ satisfies

$$d(D(t)X(t)) = \varDelta(t)\sigma(t)D(t)S(t)\widetilde{B}(t),$$

so to construct a portfolio process with value $X(t) = V(t), t \in [0, T]$, it is enough to take $X(0) = V(0)$ and choose $\varDelta(t)$ to achieve

$$d(D(t)X(t)) = d(D(t)V(t)) \Longleftrightarrow \varDelta(t)\sigma(t)D(t)S(t)\widetilde{B}(t) = \lambda(t),$$

the latter being equivalent to

$$\varDelta(t) = \frac{\lambda(t)}{\sigma(t)D(t)S(t)}. \tag{27}$$

Starting with the initial investment $V(0) = \widetilde{\mathbb{E}}[D(T)V(T)]$ and trading so as to keep $\varDelta(t), t \in [0, T]$ shares of stock will terminate in time T to a portfolio priced at $V(T)$, whichever the stock price development.

Formula (27) presumes that the volatility coefficient $\sigma(t)$ is nonzero. Under the assumption $\sigma(t) \neq 0, t \in [0, T]$, the market comprised of two instruments – stock and bank investments – is *complete*, meaning that every option with $\mathcal{F}_T$-measurable payoff at maturity T can be hedged by a self-financing stock-money portfolio.