3 Ito formula and processes

3.1 Ito formula

Let f be a differentiable function. If g is another differentiable function, we have by the chain rule

$$\frac{d}{dt}f(g(t)) = f'(g(t))g'(t),$$

which in the differential notation is written as

$$d(f(g(t)) = f'(g(t))dg(t).$$

This cannot be applied if we take for g the BM, because $B(t)$ is not differentiable. Due to the nontrivial quadratic variation, the formula has an extra term

$$d(f(B(t)) = f'(B(t))dB(t) + \frac{1}{2}f''(B(t))dt. \tag{11}$$

The exact meaning of this differential formula is

$$f(B(t)) - f(B(0)) = \int_0^t f'(B(u))dB(u) + \frac{1}{2}\int_0^t f''(B(u))du, \tag{12}$$

where the first integral should be understood as the stochastic integral.

For instance,

$$d(B^2(t)) = 2B(t)dB(t) + dt,$$

which means that

$$B^2(t) = 2\int_0^t B(u)dB(u) + t,$$

in accord with our computation of $\int_0^t B(u)dB(u)$.

The extra term comes from the Taylor formula

$$f(B(t+\delta) - B(t)) =$$
$$f'(B(t))(B(t+\delta) - B(t)) + \frac{1}{2}f''(B(t))(B(t+\delta) - B(t))^2 + o((B(t+\delta) - B(t))^2),$$

which is applicable by the continuity of BM: when $\delta \to 0$ also $(B(t+\delta) - B(t)) \to 0$. Symbol $o(\cdots)$ stays for a term of the order smaller than $\cdots$. In the mean-square sense, $(B(t+\delta) - B(t))^2 \sim \delta$ (or, heuristically, $(dB(t))^2 = dt$). Thus we have two terms up to a $o(\delta)$ remainder.

More generally, let $f(t,x)$ be a continuously differentiable function of t and x. The Ito formula in differential form is

$$df(t,B(t)) = f_t(t,B(t))dt + f_x(t,B(t))dB(t) + \frac{1}{2}f_{xx}(t,B(t))dt, \tag{13}$$

where f_x, f_t, f_{xx} denote the partial derivatives. The rigorous meaning of this is the form involving a stochastic integral:

Theorem 3.1. *For continuously differentiable* $f(t,x)$

$$f(t,B(t))-f(0,B(0))=\int_0^t f_t(u,B(u))du+\int_0^t f_x(u,B(u))dB(u)+\frac{1}{2}\int_0^t f_{xx}(u,B(u))du, \tag{14}$$

Proof. Let us use T as the upper bound of integration. Split $[0,T]$ by partition Π with points $0=t_0<t_1<\cdots<t_n=T$. Taylor's formula tells us that for small δ,ϵ

$$f(t+\delta,x+\epsilon)-f(t,x)=f_t(t,x)\delta+f_x(t,x)\epsilon+\frac{1}{2}f_{xx}(t,x)\epsilon^2+f_{tx}(t,x)\delta\epsilon+\frac{1}{2}f_{tt}(t,x)\delta^2+\cdots,$$

where $\cdots$ stays for 'higher order terms'. Taking $B(t_j)$ for x_j we calculate

$$f(T,B(T))-f(0,B(0))=\sum_{j=1}^{n-1}[f(t_{j+1},B(t_{j+1}))-f(t_j,B(t_j))]=$$

$$\sum_{j=1}^{n-1}f_t(t_j,B(t_j)(t_{j+1}-t_j)+\sum_{j=1}^{n-1}f_x(t_j,B(t_j))(B(t_{j+1})-B(t_j))$$

$$+\frac{1}{2}\sum_{j=1}^{n-1}f_{xx}(t_j,B(t_j))(B(t_{j+1})-B(t_j))^2+\sum_{j=1}^{n-1}f_{tx}(t_j,B(t_j))(B(t_{j+1})-B(t_j))(t_{j+1}-t_j)$$

$$+\frac{1}{2}\sum_{j=1}^{n-1}f_{tt}(t_j,B(t_j))(t_{j+1}-t_j)^2+\cdots$$

Sending the mesh size $\Delta\to 0$, the first sum in the RHS converges to $\int_0^T f_t(t,B(t))dt$, the second contributes stochastic integral $\int_0^T f_x(t,B(t))dB(t)$. The third sum yields in the limit $\frac{1}{2}\int_0^T f_{xx}(t,B(t))dt$ in the view of $(B(t_{j+1})-B(t_j))^2\sim(t_{j+1}-t_j)$, and other terms vanish for a similar reason.

□

Example 3.2. Geometric BM, with constant μ,σ

$$S(t)=S(0)\exp(\mu t+\sigma B(t)).$$

Applying the Ito formula we arrive at

$$dS(t)=S(t)[(\mu+\frac{1}{2}\sigma^2)dt+\sigma dB(t)].$$

This is an example of a *stochastic differential equation*.

3.2 Ito (drift-diffusion) processes

Let $(B(t),t\geq 0)$ be a BM with filtration $(\mathcal{F}_t,t\geq 0)$.

Definition 3.3. Suppose $(\mu(t), t \geq 0)$ and $(\sigma(t), t \geq 0)$ be two stochastic processes adapted to $(\mathcal{F}_t, t \geq 0)$. An Ito process is a stochastic process of the form

$$X(t) = X(0) + \int_0^t \sigma(u) dB(u) + \int_0^t \mu(u) du,$$

where $X(0)$ is nonrandom. The terms $\int_0^t \sigma(t) dB(t)$ and $\int_0^t \mu(t) dt$ are called diffusion and drift terms, respectively. It is also said that $X(t)$ has a stochastic differential

$$dX(t) = \mu(t) dt + \sigma(t) dB(t). \tag{15}$$

Note that $(X(t), t \geq 0)$ is adapted to the same filtration as the BM.

The formula for quadratic variation of Ito integral is readily extendible to the processes with drift term, since the quadratic variation of the drift term is zero. We have

$$\langle X \rangle(t) = \int_0^t \sigma^2(u) du,$$

which we also write as

$$(dX(t))^2 = \sigma^2(t) dt.$$

The formula can be obtained by formal squaring $dX(t) = \mu(t)dt + \sigma(t)dB(t)$ and using the familiar rules for differentials

$$(dB(t))^2 = dt, dB(t)dt = 0, (dt)^2 = 0.$$

Stochastic integral can be defined with integrators more general than the BM.

Definition 3.4. Let $(X(t), t \geq 0)$ be an Ito process, as in Definition 3.3. For another adapted process $(Y(t), t \geq 0)$ the stochastic integral with respect to the Ito process is defined as

$$\int_0^t Y(u) dX(u) = \int_0^t Y(u)\sigma(u) dB(u) + \int_0^t Y(u)\mu(u) du.$$

Formal manipulation of differentials involved in (15) leads to a more general Ito formula for $X(t)$ (generalising the case $X(t) = B(t)$)

$$df(t, X(t)) = f_t(t, X(t))dt + f_x(t, X(t))dX(t) + \frac{1}{2} f_{xx}(t, X(t))(dX(t))^2 =$$
$$f_t(t, X(t))dt + f_x(t, X(t))\sigma(t)dB(t) + f_x(t, X(t))\mu(t)dt + \frac{1}{2} f_{xx}(t, X(t))\sigma^2(t)dt,$$

which is proved using much the same technique as for the BM, and should be formally interpreted via its integral form (we omit arguments to simplify notation)

$$f(T, X(T)) - f(0, X(0)) = \int_0^T f_t dt + \int_0^T f_x \sigma dB + \int_0^T \mu dt + \frac{1}{2} \int_0^T f_{xx} \sigma^2 dt.$$

Example 3.5. Generalised geometric BM. For adapted processes $\alpha(t), \sigma(t)$ define

$$dX(t) = \sigma(t)dB(t) + \left(\alpha(t) - \frac{1}{2}\sigma^2(t)\right) dt,$$

and

$$S(t) = S(0)e^{X(t)}.$$

This can be seen as $S(t) = f(X(t))$ for $f(x) = S(0)e^x$.

Using the Ito formula

$$dS(t) = df(X(t)) = f'(X(t))dX(t) + \frac{1}{2}f''(X(t))(dX(t))^2 =$$
$$S(0)e^{X(t)}dX(t) + \frac{1}{2}S(0)e^{X(t)}(dX(t))^2 = S(t)\left(dX(t) + \frac{1}{2}(dX(t))^2\right) =$$
$$S(t)(\alpha(t)dt + \sigma(t)dB(t)).$$

This process models a stock price with (time-dependent) instantaneous mean rate of return $\alpha(t)$ and volatility $\sigma(t)$. If $\alpha = 0$ then $dS(t) = \sigma(t)S(t)dB(t)$, which is the same as

$$S(t) = S(0) + \int_0^t \sigma(u)S(u)dB(u).$$

The RHS is a constant plus Ito integral, which is a martingale. Thus

$$S(t) = S(0)\exp\left[\int_0^t \sigma(u)dB(u) - \frac{1}{2}\int_0^t \sigma^2(u)du\right]$$

is a martingale.

In particular, when σ is constant the gBM $S(t) = S(0)e^{\sigma B(t) - \sigma^2 t/2}$ is a martingale.

Theorem 3.6. *Let $\sigma(t)$ be a nonrandom function. Then the Ito integral*

$$I(t) = \int_0^u \sigma(u)dB(u)$$

has the normal distribution $\mathcal{N}\left(0,\ \int_0^t \sigma^2(u)du\right)$.

Proof. We know that $(I(t), t \geq 0)$ is a martingale, thus $\mathbb{E}I(t) = \mathbb{E}I(0) = 0$. Furthermore, $\mathrm{VarI}(t) = \int_0^t \sigma^2(u)du$ by the Ito isometry. To show the normal distribution note that for any fixed $\lambda \in \mathbb{R}$

$$X(t) = \int_0^t \lambda\sigma(u)dB(u) - \frac{1}{2}\int_0^t (\lambda\sigma^2(u))du$$

is a generalised gBM. As we have seen in the example above, $(e^{X(t)}, t \geq 0)$ is a martingale, thus has constant mean value

$$\mathbb{E}e^{X(t)} = \mathbb{E}e^{X(0)} = e^0 = 1,$$

which is the same as

$$\mathbb{E}\left(\int_0^t \lambda\sigma(u)dB(u) - \frac{1}{2}\int_0^t (\lambda\sigma^2(u))du\right) = 1,$$

The second integral is not a random variable, whence

$$\mathbb{E}e^{\lambda I(t)} = \exp\left(\frac{1}{2}\lambda^2 \int_0^t \sigma^2(u)du\right).$$

As a function of λ, the LHS is the moment generating function of $I(t)$, while the RHS is the mgf a normal distribution. □

3.3 Black-Scholes-Merton equation for the European call

Recall that a forward contract obliges the holder to buy (and the seller to sell) one share of stock at time T for price K. The value of such contract at time T is $S(T) - K$, where $S(T)$ is the market value at time T. What is the value of the forward at time $t < T$? Naturally, the value should depend on the stock price, say $S(t) = x$ at time t. Suppose a riskless bank investment has interest rate r for both borrowing and lending. The portfolio with one share and $-Ke^{-r(T-t)}$ in the bank account has value $f(t, x) := x - Ke^{-r(T-t)}$ at time t, and will yield $S(T) - (Ke^{-r(T-t)})e^{r(T-t)} = S(T) - K$ at time T, which is the payoff of the forward. The portfolio *hedges* the forward, therefore must have the same value at time t, for otherwise there would be an arbitrage opportunity. Thus the fair value of forward at time t is $f(t, S(t))$. Remarkably, this conclusion does not require any assumptions of the behaviour of the stock price process $S(t)$.

Such simple hedging is not possible for more complex contracts like options.

The (European) call option with strike K is the right to sell one share of stock at the maturity time T for K pounds. The value of the call at the maturity T is

$$c(T, S(T)) = (S(T) - K)^+. \tag{16}$$

Finding a fair price of the option $c(t, x)$ at time t when $S(t) = x$ is a more delicate matter. In the Black-Scholes-Merton theory it is assumed that the stock price follows a gBM

$$S(t) = S(0)\exp\left((\alpha - \sigma^2/2)t + \sigma B(t)\right), t \geq 0,$$

where $\sigma > 0$ is constant volatility parameter, $\alpha > 0$ is a mean rate of return, and B is a BM adapted to some filtration $(\mathcal{F}_t, t \geq 0)$. In the differential form,

$$dS(t) = \alpha S(t)dt + \sigma S(t)dB(t). \tag{17}$$

Suppose the investor can trade the stock and invest in bank, changing her positions permanently. At time t the portfolio has

(i) total value $X(t)$,

(ii) $\Delta(t)$ shares of stock, where $\Delta(t)$ is $\mathcal{F}_t$-measurable,

(iii) $X(t) - \mathit{\Delta}(t)S(t)$ deposit in the riskless bank account, yielding continuously compounded interest at rate r.

For self-financing trading strategy – neither capital influx, nor consumption – the value of portfolio in infinitesimal time dt changes as (substitute (17))

$$\begin{aligned} dX(t) = \mathit{\Delta}(t)dS(t) + r(X(t) - \mathit{\Delta}(t)S(t))dt = \\ \mathit{\Delta}(t)(\alpha S(t)dt + \sigma S(t)dB(t)) + r(X(t) - \mathit{\Delta}(t)S(t))dt = \\ rX(t)dt + (\alpha - r)\mathit{\Delta}(t)S(t)dt + \sigma\mathit{\Delta}(t)S(t)dB(t), \end{aligned}$$

where the first term is the same as return on the investment $X(t)$ in bank, the second term is 'the risk premium' – investor's advantage for investing in stocks (and not in bonds), while the last term appears due to volatility of the stock market. In the formula for dX the RHS has the term $rX(t)dt$, which will be eliminated by passing to the differential $d(e^{-rt}X(t))$ of the discounted portfolio value. To that end, first calculate the differential for the discounted stock value $e^{-rt}S(t)$ using Ito's formula,

$$\begin{aligned} d(e^{-rt}S(t)) = f_t(t, S(t))dt + f_x(t, S(t))dS(t) + \frac{1}{2}f_{xx}(t, S(t))(dS(t))^2 = \\ -re^{-rt}S(t)dt + e^{-rt}dS(t) + 0 = \\ (\alpha - r)e^{-rt}S(t)dt + \sigma e^{-rt}S(t)dB(t). \end{aligned}$$

Now, substituting dX in

$$\begin{aligned} d(e^{-rt}X(t)) = -re^{-rt}X(t)dt + e^{-rt}dX(t) = \\ \mathit{\Delta}(t)(\alpha - r)e^{-rt}S(t)dt + \mathit{\Delta}(t)\sigma e^{-rt}S(t)dB(t) = \\ \mathit{\Delta}(t)d(e^{-rt}S(t)) \end{aligned}$$

the term $rX(t)dt$ has cancelled, and the change in the discounted portfolio value is expressed through the discounted stock price.

Let $c(t, S(t))$ be the fair price of the call option, that is the value of a self-financing hedging portfolio, where $c(t, x)$ is a function of two variables. The differential $dc(t, S(t))$ is computed as (to simplify the notation we write c for $c(t, S(t))$)

$$dc = c_t dt + c_x dS(t) + \frac{1}{2}c_{xx}(dS(t))^2 = c_t dt + c_x(\alpha S(t)dt + \sigma S(t)dB(t)) + \frac{1}{2}c_{xx}\sigma^2 S^2(t)dt,$$

and for the discounted price we get

$$\begin{aligned} d(e^{-rt}c) = -re^{-rt}cdt + e^{-rt}dc = \\ e^{-rt}(-rc + c_t + \alpha S(t)c_x + \frac{1}{2}\sigma^2 S^2(t)c_{xx})dt + e^{-rt}\sigma S(t)c_x dB(t). \end{aligned}$$

We want to have $c(t, S(t)) = X(t), t \in [0, 1]$, for the portfolio value $X(t)$. This is the same as $e^{-rt}c(t, S(t)) = e^{-rt}X(t)$, and this in turn holds if true for $t = 0$ and $d(e^{-rt}c(t, S(t))) = d(e^{-rt}X(t))$ for $t \in [0, T)$. Equating the differentials we obtain

$$\begin{aligned} \mathit{\Delta}(t)(\alpha - r)S(t)dt + \mathit{\Delta}(t)\sigma S(t)dB(t) = \\ \left(-rc + c_t + \alpha S(t)c_x + \frac{1}{2}\sigma^2 S^2(t)c_{xx}\right)dt + \sigma S(t)c_x dB(t). \end{aligned}$$

For this to hold the respective dt- and dB-terms must coincide. Thus

$$\Delta(t) = c_x(t, S(t)), \qquad t \in [0, T),$$

which is called the *delta-hedging rule*: the partial derivative in x of the call price should be equal to the number of shares. Equating the dt terms yields

$$(\alpha - r)S(t)c_x = -rc + c_t + \alpha S(t)c_x + \frac{1}{2}\sigma^2 S^2(t)c_{xx}, \qquad t \in [0, T),$$

which simplifies to the relation

$$rc(t, S(t)) = c_t(t, S(t)) + rS(t)c_x(t, S(t)) + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t, S(t)), \qquad t \in [0, T).$$

For this to hold the function $c(t, x)$ must satisfy the equation

$$rc(t, x) = c_t(t, x) + rS(t)c_x(t, x) + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t, x), \qquad t \in [0, T), \tag{18}$$

known as the *Black-Scholes-Merton partial differential equation*. The equation is a PDE in the classical sense, its solution is a function $c(t, x)$ which being substituted in (18) along with its partial derivatives yields an identity in the variables $t \in [0, T)$ and $x \geq 0$.

From (16) the call price satisfies the terminal condition

$$c(T, x) = (x - K)^+.$$

To solve (18) backwards in time with this terminal value, we also need boundary conditions at $x = 0$ and $x = \infty$. For $x = 0$ (18) becomes an ODE $c_t(t, 0) = rc(t, 0)$ with exponential solution $c(t, 0) = e^{rt}c(0, 0)$, and taking $t = T$ we obtain $c(0, 0) = c(T, 0) = 0$, whence, quite expectedly, $c(t, 0) = 0$. The function $c(t, x)$ has no limiting value as $x \to \infty$, but we can specify how it grows (up to a vanishing remainder term) by noting that for large $S(t) = x$ the probability of the event $S(T) > K$ is close to 1, and if the event occurs the call pays the same amount $S(T) - K$ as the forward contract with delivery price K. Thus for every fixed t and x very large the call option is almost the same as the forward contract, so

$$\lim_{x \to \infty} [c(t, x) - (x - e^{-r(T-t)}K)] = 0. \tag{19}$$

The solution to the equation (18) with the terminal condition $c(T, x) = (x - K)^+$ and the boundary conditions (19) and $c(t, 0) = 0$ is given by the BSM formula

$$c(t, x) = x\Phi(d_+(T - t, x)) - Ke^{-r(T-t)}\Phi(d_-(T - t, x)), \quad 0 \leq t < T, \tag{20}$$

where

$$d_\pm(\tau, x) = \frac{1}{\sigma\sqrt{\tau}}\left[\log\frac{x}{K} + \left(r \pm \frac{\sigma^2}{2}\right)\tau\right],$$

and

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-u^2/2}du.$$

The RHS of (20) is in fact a function of the residual time $\tau = T - t$ to maturity. The BSM formula does not specify explicitly $c(T, x)$ or $c(t, 0)$ (because $d_\pm$ is then undefined), but for (20) the limits are $\lim_{t\to T} c(t, x) = (x - K)^+$ and $\lim_{x\to 0} c(t, x) = 0$ as can be checked.

It is important to note that $c(t, x)$ does not depend on α, the mean rate of return from investment in the stock. The option price $c(0, x)$ is not the expected discounted payoff of the call under the 'market measure' (probability law) which drives the gBM S. Comparing (20) with Exercise 7 (Sheet 1) we see that $c(0, x) = \mathbb{E}[e^{-rT}(\widetilde{S}(T) - K)^+]$, where $\widetilde{S}(t) = xe^{(r-\sigma^2/2)t+\sigma B(t)}$ is another gBM with drift parameter r and $\widetilde{S}(0) = x$. In the same way, $c(t, y) = \mathbb{E}[e^{-r(T-t)}(\widetilde{S}(T) - K)^+|\widetilde{S}(t) = y]$ is the conditional discounted expected payoff, as if the price of stock were the gBM $\widetilde{S}$ with mean rate of return equal to the return rate r on investment in the money market. The switch from the 'market measure' to a stock price process which, on the average, cannot beat the money market underlies the principle of risk-neutral valuation.

Recall that the European put option with strike T pays at maturity $p(T, S(T)) = (K - S(T))^+$. Pricing the put option can be derived from that of the call option by means of the put-call parity

$$x - e^{-r(T-t)}K = c(t, x) - p(t, x),$$

which says that a long position in the call and a short position in the put pay the same as the forward contract. Plugging (18) yields

$$p(t, x) = Ke^{-r(T-t)}\Phi(-d_-(T - t, x)) - x\Phi(-d_+(T - t, x)).$$

Of course, the pricing of the put option could be derived also by directly solving (18) with the terminal conditon $p(T, S(T)) = (K - S(T))^+$ and the boundary conditions $p(t, 0) = K,\ p(t, \infty) = 0$.