Stochastic Calculus and Black-Scholes Theory MTH772P
Exercises – Sheet 4 (solutions)

1. Let $\tau = \min\{t : B(t) = a, \text{ or } B(t) = -b\}$ for $a, b > 0$. Note that $B(t) \notin [-b, a]$ implies $\tau < t$. Using this show that $\tau < \infty$ a.s. (that is $\mathbb{P}(\tau < \infty) = 1$).

Solution. If $B(t) > a$ then by continuity $B(s) = a$ for some $s < t$, hence $\tau \le s < t$. Similarly for $B(t) < -b$. Using the scaling property $B(t) =_d t^{1/2}B(1)$,

$$\mathbb{P}(B(t) \notin [-b, a]) = \mathbb{P}(B(1) \notin [-bt^{-1/2}, at^{-1/2}]) = \frac{1}{\sqrt{2\pi}} \int_{x \notin [-bt^{-1/2}, at^{-1/2}]} e^{-x^2/2} dx,$$

which is arbitrarily close to 1 for large t. Hence $\mathbb{P}(\tau > t) \le \mathbb{P}(B(t) \notin [-b, a]) \to 0$ as $t \to \infty$, so $\tau < \infty$ a.s.

2. Show that for every fixed $t > 0$ and $x \in \mathbb{R}$ there exists stopping time τ such that $\tau > t$ and $B(\tau) = x$.

Solution. Just take $\tau = \min\{s : s > t, B(s) = x\}$. By the Markov property $\widetilde{B}(u) := B(t + u) - B(t)$ is a BM independent of $\mathcal{F}_t$. Given $B(t) = y$, τ has the same distribution as the first passage time τ_{x-y}, which is finite almost surely (see the lectures).

3. Let $\widetilde{B}(t) = t\,B(1/t)$. Show that $\widetilde{B}$ is a BM by following the steps:

(i) $\widetilde{B}$ is a Gaussian process,

(ii) $\widetilde{B}$ has the mean value and covariance functions like that of a BM,

(iii) $\widetilde{B}$ has continuous paths. Check continuity at $t = 0$ using the law of large numbers.

Solution. The BM is the unique Gaussian process with continuous paths and the covariance function $\rho(s, t_= s \wedge t$.

The vector $t_1\widetilde{B}(1/t_1), \ldots, t_k\widetilde{B}(1/t_k)$ has multivariate normal distribution, because it is a linear transform of $\widetilde{B}(1/t_1), \ldots, \widetilde{B}(1/t_k)$, which is normal because the BM is Gaussian.

Calculating the covariance function: $\text{Cov}(\widetilde{B}(s), \widetilde{B}(t)) = \text{Cov}(sB(1/s), tB(1/t)) = st\text{Cov}(B(1/s), B(1/t)) = st(1/s \wedge 1/t) = s \wedge t$.

It remains to check the continuity. The only troublesome point is $t = 0$. We need to show that $tB(1/t) \to 0$ as $t \to 0$. Set $u = 1/t$. By the strong law of large numbers $B(\lfloor u \rfloor)/u \to 0$ for $u \to \infty$ since $B(k)$ is a sum of k iid increments $B(i) - B(i - 1)$ with mean zero.

Let $k = \lfloor u \rfloor$. Then $|B(u) - B(k)|/u < \xi_k$ where $\xi_k = \max_{u \in [k, k+1]} |B(u) - B(k)|$. We have $\mathbb{E}\xi_k < \infty$, because $\max_{s \in [0,1]} B(s) =_d |B(1)|$. The random variables ξ_k's are iid. From this $\lim_{k \to \infty} \xi_k/k = 0$, or $(B(u) - B(\lfloor u \rfloor))/u \to 0$ as $u \to \infty$. Thus $B(u)/u = [B(\lfloor u \rfloor) + (B(u) - B(\lfloor u \rfloor)]/u \to 0$ as $u \to \infty$.

4. Find for which values σ and μ the geometric Brownian motion

$$S(t) = \exp\left(\mu B(t) - \frac{\sigma^2}{2}t\right), \; t \ge 0,$$

is a martingale.

Solution. Calculating the Ito differential,

$$dS = \mu S dB - \frac{\sigma^2}{2} S dt + \frac{\mu^2}{2} S dt = \mu S dB + \left(\frac{\mu^2}{2} - \frac{\sigma^2}{2} \right) S dt.$$

For S to be a martingale the 'dt' term must be zero, so the condition is $\mu^2 = \sigma^2$, or $|\mu| = |\sigma|$.