Stochastic Calculus and Black-Scholes Theory MTH772P
Exercises – Sheet 4

1. Let $\tau = \min\{t : B(t) = a, \text{ or } B(t) = -b\}$ for $a, b > 0$. Note that $B(t) \notin [-b, a]$ implies $\tau < t$. Using this show that $\tau < \infty$ a.s. (that is $\mathbb{P}(\tau < \infty) = 1$).

2. Show that for every fixed $t > 0$ and $x \in \mathbb{R}$ there exists stopping time τ such that $\tau > t$ and $B(\tau) = x$.

3. Let $\widetilde{B}(t) = t\, B(1/t)$. Show that $\widetilde{B}$ is a BM by following the steps:

   (i) $\widetilde{B}$ is a Gaussian process,

   (ii) $\widetilde{B}$ has the mean value and covariance functions like that of a BM,

   (iii) $\widetilde{B}$ has continuous paths. Check continuity at $t = 0$ using the law of large numbers.

4. Find for which values σ and μ the geometric Brownian motion

$$S(t) = \exp\left(\mu B(t) - \frac{\sigma^2}{2}\, t\right), \; t \geq 0,$$

is a martingale.