Stochastic Calculus and Black-Scholes Theory MTH772P
Solutions Exercises – Sheet 3

1. A k-dimensional Brownian motion is a random vector-function $(B_1(t), \dots, B_k(t))$, where B_i are k independent (standard) BM. For smooth function f of the k-dimensional BM the stochastic differential is

$$df(B_1(t), \dots, B_k(t)) = \sum_{i=1}^{k} f_{x_i}(B_1(t), \dots, B_k(t))dB_i(t) + \frac{1}{2}\sum_{i=1}^{k} f_{x_i,x_i}(B_1(t), \dots, B_k(t))dt.$$

(This follows by taking the Taylor expansion up to the second order terms and using the rules $(dB_i)^2 = dt, \quad dB_i dB_j = 0$ for $i \neq j$).

The *Bessel process* with parameter k is defined as $X(t) = \sqrt{\sum_{i=1}^{k} B_i^2(t)}$, which is the 'radial part' of the k-dimensional BM.

(a) Show that X satisfies the stochastic differential equation

$$dX(t) = \sum_{i=1}^{k} \frac{B_i(t)}{X(t)} dB_i(t) + \frac{n-1}{X(t)} dt$$

(b) Show that X satisfies the stochastic differential equation with single BM B

$$dX(t) = dB(t) + \frac{n-1}{X(t)} dt.$$

Hint: use Lévy's theorem which states that a continuous martingale $M(t)$ with quadratic variation $\langle M \rangle(t) = t$ is a BM.

(a) Let $f = \sqrt{\sum_{i=1}^{n} x_i^2}$. Then

$$f_{x_i} = \frac{x_i}{f}, \qquad f_{x_i,x_i} = \frac{-x_i^2}{f^3} + \frac{1}{f}.$$

So we have (we write X for $X(t)$ etc)

$$dX = \sum_{i=1}^{n} \frac{B_i dB_i}{X} + \sum_{i=1}^{n} \left(\frac{B_i^2}{X^3} + \frac{1}{X} \right) dt = \sum_{i=1}^{n} \frac{B_i dB_i}{X} - \frac{X^2}{X^3} + \frac{n}{X} = \sum_{i=1}^{n} \frac{B_i(t)}{X(t)} dB_i + \frac{n-1}{X(t)} dt.$$

With respect to the filtration generated by $B_1, \dots, B_n$ the process

$$\frac{B_i(t)}{X(t)} dB_i$$

is a martingale with continuous paths, hence also the process Y with

$$dY(t) = \sum_{i=1}^{n} \frac{B_i(t)}{X(t)} dB_i$$

is a martingale with continuous paths. Squaring and using the stated rule for $dB_i dB_j$ we get

$$(dY)^2 = \sum_{i=1}^{n} \frac{B_i^2}{X^2} dt = dt.$$

It follows that the quadratic variation is $\langle Y\rangle(t) = t$. By Lévys theorem Y is a BM, hence X satisfies

$$dX(t) = dY(t) + \frac{n-1}{X(t)}dt,$$

with Y a BM.

2. Let $X(t) = B(t) + t\mu$ be a BM with dift. Use Girsanov's theorem to derive the joint density of $X(t_1), X(t_2), X(t_3)$ for $t_1 < t_2 < t_3$.

Let $f_\mu(x_1, x_2, x_3)$ be the joint density of of $X(t_1), X(t_2), X(t_3)$ when the drift coefficient is μ.

Suppose first that $\mu = 0$; then we deal with the BM. We know that $X(t_3) - X(t_2), X(t_2) - X(t_1), X(t_1)$ are independent mean-value normal random variables, with variances $t_3 - t_2, t_2 - t_1, t_1$. The Jacobian for transition from $X(t_1), X(t_2), X(t_3)$ to $X(t_1), X(t_2) - X(t_1), X(t_3) - X(t_2)$ equals 1, because the transition matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

has determinant 1. Thus the joint density of $X(t_1), X(t_2), X(t_3)$ (assuming $\mu = 0$) is

$$f_0(x_1, x_2, x_3) = ce^{-x_1^2/2}e^{-(x_2-x_1)^2/(t_2-t_1)}e^{-(x_3-x_2)^2/(t_3-t_2)}$$

where

$$c = \frac{1}{(2\pi)^{3/2}\sqrt{t_1(t_2-t_1)(t_3-t_2)}}.$$

Changing the measure from $\widetilde{\mathbb{P}}$ to $\mathbb{P}$ with the Radon-Nikodym derivative $d\widetilde{\mathbb{P}}/d\mathbb{P} = Z$, where

$$Z = e^{-\mu B(t_3) - \mu^2 t_3/2} = e^{-\mu(X(t_3) - \mu t_3) - \mu^2 t_3/2},$$

we achieve that $(X(t), t \in [0, t_3])$ is a BM (Girsanov's theorem). Conversely, to pass from $\mathbb{P}$ to $\widetilde{\mathbb{P}}$ we need the Radon-Nikodym derivative Z^{-1}. To obtain f_μ we just need to multiply f_0 with Z^{-1}, the latter considered as a function of x_3

$$f_\mu(x_1, x_2, x_3) = f_0(x_1, x_2, x_3)\exp(\mu(x_3 - \mu t_3) + \mu^2 t_3/2) = \\ f_0(x_1, x_2, x_3)\exp(\mu x_3 - \mu^2 t_3/2)$$

Re-combining the exponents

$$-\frac{x_1^2}{2t_1} - \frac{(x_2-x_1)^2}{2(t_2-t_1)} - \frac{(x_3-x_2)^2}{2(t_3-t_3)} + \mu x_3 - \mu^2 t_3/2 = \\ -\frac{(x_1-\mu t_1)^2}{2t_1} - \frac{(x_2 - x_1 - \mu(t_2-t_1))^2}{2(t_2-t_1)} - \frac{(x_3 - x_2 - \mu(t_3-t_2))^2}{2(t_3-t_2)}$$

we obtain

$$f_\mu(x_1, x_2, x_3) = \\ c\exp\left(-\frac{(x_1-\mu t_1)^2}{2t_1} - \frac{(x_2 - x_1 - \mu(t_2-t_1))^2}{2(t_2-t_1)} - \frac{(x_3 - x_2 - \mu(t_3-t_2))^2}{2(t_3-t_2)}\right)$$

which can be written as

$$c \prod_{j=0}^{2} \exp\left(-\frac{(x_{j+1} - x_j - \mu(t_{j+1} - t_j))^2}{2(t_{j+1} - t_j)} \right),$$

where $t_0 = x_0 = 0$.

Comment: the same could be derived more directly by inspecting the increments of the BM with drift.

3. For which constant σ, μ the process $S(t) = \exp(\sigma B(t) + \mu t)$ is a martingale?

We compute using Ito formula

$$dS = \sigma S dB + \mu S dt + \frac{1}{2}\sigma^2 S dt.$$

The dt terms vanishes if $\mu = \sigma^2/2$, hence under this condition we get a martingale.

4. Show that the *Ornstein-Uhlenbeck process*

$$X(t) = e^{-\alpha t}x + e^{-\alpha t}\int_0^t e^{\alpha s} dB(s)$$

satisfies the SDE

$$dX(t) = -\alpha X(t)dt + dB(t).$$

Compute the differential

$$dX = -\alpha e^{-\alpha t}x - \alpha e^{-\alpha t}\int_0^t e^{\alpha s} dB(s) + e^{-\alpha t}e^{\alpha t}dB(t) = -\alpha X(t)dt + dB(t).$$

5. Calculate the quadratic variation $\langle X \rangle(t)$ for $X(t) = e^{B^2(t)}$.

We have $dX = 2Be^{B^2}dB$ plus a dt term, so

$$\langle X \rangle(T) = \mathbb{E}\int_0^T 4B^2(t)e^{2B^2(t)}dt = \int_0^t \mathbb{E}[4B^2(t)e^{2B^2(t)}]dt.$$

The expectation is

$$\mathbb{E}[4B^2(u)e^{2B^2(u)}] = \frac{1}{\sqrt{2\pi T}}\int_{-\infty}^{\infty} e^{-x^2/(2t)}4x^2e^{2x^2}dx.$$

If $t > 1/4$ the integral is infinite, therefore $\langle X \rangle(T) = \infty$ for $T \geq 1/4$. For $T < 1/4$

$$\langle X \rangle(T) = \int_0^T \frac{4}{t^{1/2}(t^{-1} - 2)^{3/2}}dt$$