Stochastic Calculus and Black-Scholes Theory MTH772P
Exercises – Sheet 3

1. A k-dimensional Brownian motion is a random vector-function $(B_1(t), \ldots, B_k(t))$, where B_i are k independent (standard) BM. For smooth function f of the k-dimensional BM the stochastic differential is

$$df(B_1(t), \ldots, B_k(t)) = \sum_{i=1}^{k} f_{x_i}(B_1(t), \ldots, B_k(t))dB_i(t) + \frac{1}{2}\sum_{i=1}^{k} f_{x_i,x_i}(B_1(t), \ldots, B_k(t))dt.$$

(This follows by taking the Taylor expansion up to the second order terms and using the rules $(dB_i)^2 = dt, \quad dB_i dB_j = 0$ for $i \neq j$).

The *Bessel process* with parameter k is defined as $X(t) = \sqrt{\sum_{i=1}^{k} B_i^2(t)}$, which is the 'radial part' of the k-dimensional BM.

(a) Show that X satisfies the stochastic differential equation

$$dX(t) = \sum_{i=1}^{k} \frac{B_i(t)}{X(t)} dB_i(t) + \frac{n-1}{X(t)} dt$$

(b) Show that X satisfies the stochastic differential equation with single BM B

$$dX(t) = dB(t) + \frac{n-1}{X(t)} dt.$$

Hint: use Lévy's theorem which states that a continuous martingale $M(t)$ with quadratic variation $\langle M \rangle(t) = t$ is a BM.

2. Let $X(t) = B(t) + t\mu$ be a BM with dift. Use Girsanov's theorem to derive the joint density of $X(t_1), X(t_2), X(t_3)$ for $t_1 < t_2 < t_3$.

3. For which constant σ, μ the process $S(t) = \exp(\sigma B(t) + \mu t)$ is a martingale?

4. Show that the *Ornstein-Uhlenbeck process*

$$X(t) = e^{-\alpha t} x + e^{-\alpha t} \int_0^t e^{\alpha s} dB(s)$$

satisfies the SDE

$$dX(t) = -\alpha X(t) dt + dB(t).$$

5. Calculate the quadratic variation $\langle X \rangle(t)$ for $X(t) = e^{B^2(t)}$.