# Stochastic Calculus and Black-Scholes Theory MTH772P
## Exercises – Sheet 2 (solutions)

1. Which of the following three processes is a martingale?

(i) $\int_0^t B(u)du - tB(t),\ t \geq 0,$

(ii) $\int_0^t B(u)du - B^3(t)/3,\ t \geq 0,$

(iii) $B^3(t)/3 - tB(t),\ t \geq 0.$

The fastest way is calculating the differentials using Ito's formula

$$df(t, B(t)) = f_x(t, B)dB(t) + f_t(t, B(t))dt + \frac{1}{2}f_{xx}(t, B(t))$$

If it turns that $dX(t) = \alpha(t, B(t))dB(t)$ then

$$X(t) - X(0) = \int_0^t \alpha(u, B(u)dB(u),$$

is an Ito integral, thus $(X(t), t \geq 0)$ is a martingale.

(i) $d(\int_0^t B(u)du - tB(t)) = B(t)dt - B(t)dt - tdB(t) = tdB(t)$ there is no drift '$dt$' term, therefore $X(t) = \int_0^t B(u)du - tB(t)$ is a martingale.

(ii) similar

(iii) For $X(t) = B^3(t)/3 - tB(t)$ we have (we omit dependence on $t$)
$dX = 3B^2 \cdot \frac{1}{3}dB + \frac{1}{2} \cdot 6 \cdot \frac{1}{3}Bdt - dt \cdot B - t \cdot dB = (B^2 - t)dB$. No drift, hence a martingale.

2. Calculate the stochastic differentials

(i) $d\sin(t\,B(t)),$

(ii) $dB^4(t),$

Using (iii) express the stochastic integral $\int_0^t B^3(u)dB(u)$ in terms of $B(t)$ and of an ordinary integral involving the BM.

(i)

$$d\sin(t\,B(t)) = t\cos(t\,B)dB + B\cos(t, B)dt - \frac{1}{2}t^2\sin(t\,B)dt.$$

(ii) similar

3. Lévy's theorem characterises the BM as a continuous martingale with qudratic variation on $[0, t]$ equal $t$. Use the theorem to prove the following result. Let $(B_1(t), t \geq 0)$, and $(B_2(t), t \geq 0)$ be two independent BM, and let $|\rho| \leq 1$. Then

$$X(t) = \rho B_1(t) + \sqrt{1 - \rho^2}B_2(t)$$

is a BM.

Obviously, $X(0) = 0$. The paths of $X$ are continuous since each term $\rho B_1(t)$ and $\sqrt{1-\rho^2}B_2(t)$ is continuous (by the continuity of BM). Each of the terms is a martingale, hence the sum is a martingale:

$$\mathbb{E}(\rho B_1(t) + \sqrt{1-\rho^2}B_2(t)|\mathcal{F}_t) =$$
$$\mathbb{E}(\rho B_1(t)|\mathcal{F}_s) + \mathbb{E}(\sqrt{1-\rho^2}B_2(t)|\mathcal{F}_s) = \rho B_1(s) + \sqrt{1-\rho^2}B_2(s).$$

The quadratic variation of $\rho B_1(t)$ is $\langle cB_1(t)\rangle = c^t$. By independence of the terms, the quadratic of $X$ is $\langle X\rangle(t) = \rho^2 t + (1-\rho^2)t = t$. Using Levy's theorem $X$ is a BM.

4. Consider a discrete-time binomial model, in which $S(0) = 1$ and $S(t+1) = 2S(t)$ or $S(t+1) = S(t)/2$ for $t = 0, 1, \ldots$. A bank account earns interest $r = 1/4$ per unit time.

(i) Construct the risk-neutral measure.

(ii) Price the option which pays $S^2(T)$ at expiration $T = 2$. Calculate explicitly the hedging portfolio.

(iii) Suppose in the unit time the stock doubles with probability 2/3 and halves with probability 1/3. For $T = 2$ determine the Radon-Nikodym derivative $Z$ with respect to the RN measure from (i). Note that $Z$ is a function of the path (like 'up,up' , 'down,up', etc).

(i)The probability for 'up' (doubling) is $(1 + 0.25 - 0.5)/(2 - 0.5) = 0.5$; same for 'down'.

(ii) The risk-neutral probabilities of the values of $S^2(2) = 16.0; 1.0; 0.0625$ are $0.25; 0.5; 0.25$, respectively. The price of the option is the discounted expected value

$$V = (16.0 \cdot 0.25 + 1.0 \cdot 0.5 + 0.0625 \cdot 0.25)/(1 + 0.25)^2 = 2.89$$

The hedging portfolio depends on the first up or down. If 'up', the number of shares is $\Delta = (4^2 - 1^2)/(4 - 1) = 5$, and the bank investment $b$ is to achieve $(1 + r)b + 4 \cdot 5 = 16$, so $b = -4/1.25 = 3.2$. Similarly for 'down'.

(iii) For instance $Z(\text{up}, \text{up}) = \widetilde{\mathbb{P}}(\text{up}, \text{up})/\mathbb{P}(\text{up}, \text{up}) = (2/3)^2/(1/2)^2 = 0.1111$

5. Let $\xi$ be standard normal. Consider a new measure with the Radon-Nikodym derivative $Z = d\widetilde{\mathbb{P}}/d\mathbb{P}$ being $Z = \exp(-\theta\xi - \theta^2/2)$. Use the moment generating function to determine the distribution of $\xi$ under $\widetilde{\mathbb{P}}$. (See Example 4.1 in the lecture notes)

The moment generating function (mgf) of $\mathcal{N}(\theta, 1)$ is $\phi(t) = \exp(\theta t + t^2/2)$, but all we actually need is the definition of mgf. Under $\widetilde{\mathbb{P}}$

$$\phi(t) = \widetilde{\mathbb{E}}\xi^t = \mathbb{E}\xi^t e^{\theta\xi - \theta^2/2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} x^t e^{-\theta x - \theta^2/2} dx =$$
$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x+\theta)^2/2} x^t dx.$$

This is the mgf of $\mathcal{N}(-\theta, 1)$.

6. For continuously differentiable function $f$ show, applying the Ito formula, that

$$\int_0^t f(u)dB(u) = f(t)B(t) - \int_0^t f'(u)B(u)du.$$

Use this integration by parts formula in the case $f(t) = t$ to determine the distribution of the ordinary integral $\int_0^t B(u)du$. (In the tutorial class 31/01 this was obtained directly from the definitions of Riemann integral and BM.)

We have equality for $t = 0$. Compute the differentials:

$$f(t)dB(t) = f'(t)B(t)dt + f(t)dB(t) - f'(t)B(t)dt = f(t)dB(t).$$

Since the differentials and the initial values are the same, we have the required identity.

Take $f(t) = t$, so $f'(t) = 1$. From the above

$$tB(t) - \int_0^t B(u)du = \int_0^t udB(u).$$

Re-write as

$$\int_0^t B(u)du = tB(t) - \int_0^t udB(u) = \int_0^t (t-u)dB(u).$$

This is the Ito integral with nonrandom integrand, hence the integral has normal distribution with mean zero (as Ito integrals are martingales) and the variance computed by the Ito isometry as

$$\int_0^t (t-u)^2 du = \frac{t^3}{3}.$$