# Stochastic Calculus and Black-Scholes Theory MTH772P
## Exercises – Sheet 2

1. Which of the following three processes is a martingale?

(i) $\int_0^t B(u)du - tB(t),\ t \geq 0,$

(ii) $\int_0^t B(u)du - B^3(t)/3,\ t \geq 0,$

(iii) $B^3(t)/3 - tB(t),\ t \geq 0.$

2. Calculate the stochastic differentials

(i) $d\sin(t\,B(t)),$

(ii) $dB^4(t),$

Using (ii) express the stochastic integral $\int_0^t B^3(u)dB(u)$ in terms of $B(t)$ and of an ordinary integral involving the BM.

3. Lévy's theorem characterises the BM as a continuous martingale with qudratic variation on $[0,t]$ equal $t$. Use the theorem to prove the following result. Let $(B_1(t), t \geq 0)$, and $(B_2(t), t \geq 0)$ be two independent BM, and let $|\rho| \leq 1$. Then

$$B(t) = \rho B_1(t) + \sqrt{1-\rho^2}B_2(t)$$

is a BM.

4. Consider a discrete-time binomial model, in which $S(0) = 1$ and $S(t+1) = 2S(t)$ or $S(t+1) = S(t)/2$ for $t = 0, 1, \ldots$. A bank account earns interest $r = 1/4$ per unit time.

(i) Construct the risk-neutral measure.

(ii) Price the option which pays $S^2(T)$ at expiration $T = 2$. Calculate explicitly the hedging portfolio.

(iii) Suppose in the unit time the stock doubles with probability 2/3 and halves with probability 1/3. For $T = 2$ determine the Radon-Nikodym derivative $Z$ with respect to the RN measure from (i). Note that $Z$ is a function of the path (like 'up,up' , 'down,up', etc).

5. Let $\xi$ be standard normal. Consider a new measure with the Radon-Nikodym derivative $Z = d\widetilde{\mathbb{P}}/d\mathbb{P}$ being $Z = \exp(-\theta\xi - \theta^2/2)$. Use the moment generating function to determine the distribution of $\xi$ under $\widetilde{\mathbb{P}}$. (See Example 4.1 in the lecture notes)

6. For continuously differentiable function $f$ show, applying the Ito formula, that

$$\int_0^t f(u)dB(u) = f(t)B(t) - \int_0^t f'(u)B(u)du.$$

Use this integration by parts formula in the case $f(t) = t$ to determine the distribution of the ordinary integral $\int_0^t B(u)du$. (In the tutorial class 31/01 this was obtained directly from the definitions of Riemann integral and BM.)