Stochastic Calculus and Black-Scholes Theory MTH772P
Exercises – Sheet 1

1. For $\xi_1, \xi_2, \cdots$ i.i.d. with $\mathbb{P}(\xi_i = \pm 1) = 1/2$ define the discrete-time random walk

$$W_0 = 0, \quad W_n = \xi_1 + \ldots + \xi_n.$$

(i) Formulate and prove the property of independence of increments of $(W_n, n \geq 0)$.

(ii) Show that $(W_n, n \geq 0)$ is a discrete-time Markov chain.

(iii) Show that $(W_n, n \geq 0)$ is a discrete-time martingale.

(iv) Calculate the covariance $\mathrm{Cov}(W_i, W_j)$.

(v) Find the limit distribution of $W_{\lfloor nt \rfloor}/\sqrt{n}$ as $n \to \infty$, for $t > 0$.

Solution (i) For any integer times $0 = j_0 < j_1 < \ldots < j_k$ the increments $W_{j_1} - W_{j_0} = \xi_1 + \ldots + \xi_{j_1}, \ldots, W_{j_k} - W_{j_{k-1}} = \xi_{j_{k-1}+1} + \ldots + \xi_{j_k}$ are independent. Because $\xi_1, \xi_2, \ldots$ are independent by the assumption, the vectors $(\xi_1, \ldots, \xi_{j_1}), \ldots, (\xi_{j_{k-1}+1}, \ldots, \xi_{j_k})$ are independent, hence the increments are independent as functions of the independent random vectors.

(ii) For any possible path $w_0, \ldots, w_n$ of the random walk we have

$$\mathbb{P}(W_n = w_n | W_{n-1} = w_{n-1}, W_{n-2} = w_{n-2}, \ldots, W_0 = w_0) =$$
$$\mathbb{P}(W_n - W_{n-1} = w_n - w_{n-1} | W_{n-1} = w_{n-1}, W_{n-2} = w_{n-2}, \ldots, W_0 = w_0).$$

In this conditional probability the event $\{W_n - W_{n-1} = w_n - w_{n-1}\}$ is the same as $\{\xi_n = w_n - w_{n-1}\}$, while the event $\{W_{n-1} = w_{n-1}, W_{n-2} = w_{n-2}, \ldots, W_0 = w_0\}$ can be written in terms of $\xi_1, \ldots, \xi_{n-1}$. It follows from the independence of ξ_i's that

$$\begin{aligned} \mathbb{P}(W_n = w_n | W_{n-1} = w_{n-1}, W_{n-2} = w_{n-2}, \ldots, W_0 = w_0) &= \\ \mathbb{P}(W_n - W_{n-1} = w_n - w_{n-1}) = \mathbb{P}(W_n - W_{n-1} = w_n - w_{n-1} | W_{n-1} = w_{n-1}) &= \\ \mathbb{P}(W_n = w_n | W_{n-1} = w_{n-1}), & \end{aligned}$$

which is the Markov property for discrete-time processes.

(iii) We have using rules for conditional expectations

$$\mathbb{E}(W_{n+1} | W_1, \ldots, W_n) = \mathbb{E}(W_n + \xi_{n+1} | W_1, \ldots, W_n) =$$
$$\mathbb{E}(W_n | W_1, \ldots, W_n) + \mathbb{E}(\xi_{n+1} | W_1, \ldots, W_n) = W_n + \mathbb{E}\xi_{n+1}.$$

Note that we could also write the conditional expectation $\mathbb{E}(W_{n+1} | W_1, \ldots, W_n)$ as $\mathbb{E}(W_{n+1} | \mathcal{F}_n)$, where $\mathcal{F}_n$ is the σ-algebra generated by the events $\{W_1 = w_1, \ldots, W_n = w_n\}$ with arbitrary $w_1, \ldots, w_n$. Also, $\mathcal{F}_n$ is the σ-algebra generated by the random variables $\xi_1, \ldots, \xi_n$.

(iv) Assume $i \leq j$. Since $\mathbb{E}W_i = 0$ we have $\mathrm{Cov}(W_i, W_j) = \mathbb{E}(W_i W_j) = \mathbb{E}(W_i(W_i + (W_j - W_i)) = \mathbb{E}(W_i^2) + \mathbb{E}(W_i)\mathbb{E}(W_j - W_i) = \mathrm{Var}(W_i) = i\mathrm{Var}\xi_1 = i$. Thus $\mathrm{Cov}(W_i, W_j) = i \wedge j$ for any $i, j \geq 0$.

(v) The limit distribution of $(\xi_1 + \ldots + \xi_{\lfloor nt \rfloor})/\sqrt{\lfloor nt \rfloor}$ (for every fixed $t > 0$ and $n \to \infty$) is $\mathcal{N}(0, 1)$ by the central limit theorem. From this, the limit distribution of $W_{\lfloor nt \rfloor}/\sqrt{n}$ is $\mathcal{N}(0, t)$.

2. Let $B = (B(t), t \geq 0)$ be a standard Brownian motion. Show that the following processes are standard BM:

(i) $X(t) = B(t+s) - B(s)$, where $s \geq 0$ is constant.

(ii) $X(t) = B(ct)/\sqrt{c}$, for any $c > 0$,

(iii) $X(t) = B(1-t) - B(1)$, where $t \in [0,1]$ (this BM is defined on $[0,1]$).

Solution (i) The process has continuous paths and $X(0) = 0$. The increments of $X(t_{i+1}) - X(t_i)$ over intervals of partition $0 = t_0 < t_1 < \ldots < t_n$ are the increments of the BM over the intervals between times $s < t_1 + s < \ldots < t_n + s$, hence they are independent and $\mathcal{N}(0, (t_{i+1} - t_i))$-distributed.

(ii) $B(ct)$ is $\mathcal{N}(0, ct)$-distributed, hence $B(ct)/\sqrt{c}$ is $\mathcal{N}(0, t)$-distributed (check the mean and the variance). We have $X(0) = 0$. The increments of X over the intervals of partition $0 = t_0 < t_1 < \ldots < t_n$ are the increments of BM over the intervals of partition $0 = ct_0 < ct_1 < \ldots < ct_n$, hence the X-increments are independent.

(iii) The increments of $X(t_{j+1}) - X(t_j)$ over $0 = t_0 < t_1 < \ldots < t_n = 1$ are the increments of the BM over the intervals of partition $1 - t_n < \ldots < 1 - t_0$. The independence of increments follows. The rest is obvious.

3. For X a random variable with density function f consider the event $A = \{X \leq 0\}$.

(i) Define $\mathcal{G}$ to be the σ-algebra generated by A (i.e. the smallest σ-algebra containing event A). Write down the list of all elements of the σ-algebra $\mathcal{G}$.

(ii) In terms of integrals with density f, describe the random variable $\mathbb{E}(X^3|\mathcal{G})$.

(iii) Using the formulas you derived in (ii) show explicitly that $\mathbb{E}(\mathbb{E}(X^3|\mathcal{G})) = \mathbb{E}(X^3)$.

(iv) Make the calculations for (ii), (iii) assuming that X has $\mathcal{N}(0,1)$ distribution.

Solution (i) $\mathcal{G} = \{\varnothing, \Omega, \{X \leq 0\}, \{X > 0\}\}$.

(ii) The events $\{X \leq 0\}, \{X > 0\}$ are disjoint and their union is Ω. Hence we can write

$$\mathbb{E}(X^3|\mathcal{G}) = \mathbb{E}(X^3|X \geq 0)1(X \geq 0) + \mathbb{E}(X^3|X \geq 0)1(X < 0),$$

where $1(\cdots)$ is the indicator random variable. In particular, $\mathbb{E}(X^3|\mathcal{G})$ may take two values, depending on whether $X \geq 0$ or $X < 0$. In terms of the density, these values are

$$\mathbb{E}(X^3|X \geq 0) = \frac{\int_0^\infty x^3 f(x)dx}{\int_0^\infty f(x)dx}, \qquad \mathbb{E}(X^3|X < 0) = \frac{\int_{-\infty}^0 x^3 f(x)dx}{\int_{-\infty}^0 f(x)dx}.$$

4. Let $(\mathcal{F}_t, t \geq 0)$ be a filtration for BM. That means that $\{\omega \in \Omega : B(s) \leq x\} \in \mathcal{F}_t$ for $s \leq t$ and $x \in \mathbb{R}$, and that the increments of BM after t are independent of $\mathcal{F}_t$. For $0 < a < b < c$ show that $B(c) - B(b)$ is independent of $\mathcal{F}_a$.

Solution We have $B(c) - B(b)$ independent of $\mathcal{F}_b$, which means that any event $\{B(c) - B(a) < x\}$ is independent of any event $A \in \mathcal{F}_b$. But $\mathcal{F}_a \subset \mathcal{F}_b$, hence $B(c) - B(b)$ is also independent of $\mathcal{F}_a$.

5. (BM with drift) Let $X(t) = B(t) + t\mu$. Show that $(X(t), t \geq 0)$ is a Markov process and find its transition density. Is the process a martingale?

Solution Let $(\mathcal{F}_t, t \geq 0)$ be a filtration for the BM. The Markov property of the BM itself means that for $s < t$

$$\mathbb{E}(f(B(t))|\mathcal{F}_s) = \mathbb{E}(f(B(t))|B(s))$$

for any function f. (Note that if we take $f(x) = 1(x \leq a)$ the conditional expectation becomes the conditional probability $\mathbb{E}[1(B(t) \leq a)|\mathcal{F}_s] = \mathbb{P}[B(t) \leq a|\mathcal{F}_s])$.

Thus from the Markov property of BM

$$\mathbb{E}(f(X(t))|\mathcal{F}_s) = \mathbb{E}(f(B(t) + t\mu)|\mathcal{F}_s) = \mathbb{E}(f(B(t) + t\mu)|B(s)).$$

But conditioning on $B(s)$ is the same as conditioning on $X(s) = B(s) + s\mu$, because $X(s)$ and $B(s)$ uniquely determine one another. Hence the above becomes

$$\mathbb{E}(f(B(t) + t\mu)|B(s)) = \mathbb{E}(f(B(t) + t\mu)|X(s)) = \mathbb{E}(f(X(t))|X(s)),$$

and so

$$\mathbb{E}(f(X(t))|\mathcal{F}_s) = \mathbb{E}(f(X(t))|X(s)),$$

which is the Markov property for $(X(t), t \geq 0)$.

To compute the transition probability function of X we should reduce to the BM, and there are various equivalent ways to do that. The probability that X moves from $X(s) = x$ to some value $X(t) \leq y$ is

$$\mathbb{P}(X(t) \leq y|X(s) = x) = \mathbb{P}(B(t) + t\mu \leq y|B(s) = x - s\mu) =$$
$$\mathbb{P}(B(t) \leq y - t\mu|B(s) = x - s\mu) = \mathbb{P}(B(t) - B(s) \leq y - x - (t-s)\mu|B(s) = x - s\mu) =$$
$$\mathbb{P}(B(t) - B(s) \leq y - x - \mu(t-s)) = \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{y-x-(t-s)\mu} \exp\{-u^2/(2t-2s)\}du$$

The transition density from x to y in time $t - s$ is obtained by differentiating this (conditional distribution function) in y

$$\tilde{p}(t-s, x, y) = \frac{e^{-(y-x-(t-s)\mu)^2/(2t-2s)}}{\sqrt{2\pi(t-s)}}$$

Another possibility is to use the fact that the transition density satisfies

$$\mathbb{E}(f(X(t))|X(s) = x) = \int_{-\infty}^{\infty} \tilde{p}(t-s, x, y)f(y)dy$$

for any function f. Using Lemma 1.9 from the lecture notes

$$\mathbb{E}(f(X(t))|X(s) = x) = \mathbb{E}[f((B(t) - B(s)) + B(s) + t\mu)|B(s) = x - s\mu] =$$
$$\mathbb{E}[f((B(t) - B(s)) + x - s\mu + t\mu] = \int_{-\infty}^{\infty} p(t-s, 0, y)f(u + x - s\mu + t\mu)du =$$
$$\int_{-\infty}^{\infty} \tilde{p}(t-s, x, y)f(y)dy,$$

where the last step used change of variable $u + x - s\mu + t\mu = y$ and the transition density $p(t-s, 0, y) = \exp(-2y^2/(2t-2s))/\sqrt{2\pi(t-s)}$ of the BM.

Finally, somewhat heuristic but quick way is as follows. Process X moves from $X(s) = x$ to $X(t) \in [y, y + dy]$ when the BM moves from $B(s) = x - s\mu$ to $B(t) \in [y - t\mu, y + dy - t\mu]$. The latter is an event of probability

$$p(t - s, x - s\mu, y - t\mu)dy = p(t - s, 0, y - x - (t - s)\mu)dy.$$

Discarding dy yields the transition density $\tilde{p}(t - s, x, y) = p(t - s, 0, y - x - (t - s)\mu)$ for the process X.

6. (Geometric BM) Let $S(t) = S(0)\exp(\nu t + \sigma B(t))$, where $S(0), \sigma, \nu$ are positive constants. Show that $(S(t), t \geq 0)$ is a Markov process and find its transition density.

Solution The Markov property is shown as in Exercise 5: if $S(s) = x$ then $B(s) = (\log \frac{x}{S_0} - \nu t)/\sigma$, so we can compute $S(s)$ from $B(s)$ and vice versa. Let for shorthand $b = (\log \frac{x}{S_0} - \nu t)/\sigma$. We have

$$\mathbb{E}[f(S(t))|S(s) = x] = \mathbb{E}[f(S(t))|B(s) = b] = \mathbb{E}[f(S(0)e^{\nu t+\sigma(B(t)-B(s))+\sigma B(s)})|B(s) = b] =$$
$$\mathbb{E}[f(S(0)e^{\nu t+\sigma(B(t)-B(s))+\sigma b})] = \int_{-\infty}^{\infty} f(S(0)e^{\nu t+\sigma u+\sigma b})\frac{e^{-u^2/(2t-2s)}}{\sqrt{2\pi(t-s)}}du.$$

Using the change of variable (recall the definition of b)

$$y = S(0)e^{\nu t+\sigma u+\sigma b}, \quad u = \frac{\log(y/x) - \nu(t - s)}{\sigma}, \quad du = \frac{dy}{\sigma y}$$

the above integral becomes

$$\int_0^{\infty} f(y)\frac{\exp\left(-\frac{(\log(y/x)-\nu(t-s))^2}{2\sigma^2(t-s)}\right)}{\sigma y\sqrt{2\pi(t-s)}}dy.$$

Therefore the transition density function of $(S(t), t \geq 0)$ is

$$\hat{p}(t - s, x, y) = \frac{\exp\left(-\frac{(\log(y/x)-\nu(t-s))^2}{2\sigma^2(t-s)}\right)}{\sigma y\sqrt{2\pi(t-s)}}.$$

7. (Black-Scholes formula) Let $S(t) = S(0)\exp((r - \sigma^2/2)t + \sigma B(t))$, where $S(0), \sigma, r$ are positive constants. For $K > 0$ and $T > 0$ show that

$$\mathbb{E}[e^{-rT}(S(T) - K)^+] = S(0)\Phi(d_+(T, S(0))) - Ke^{-rT}\Phi(d_-(T, S(0))),$$

where Φ is the standard normal distribution function, and

$$d_\pm(T, S(0)) = \frac{1}{\sigma\sqrt{T}}\left(\log\frac{S(0)}{K} + (r \pm \frac{\sigma^2}{2})T\right).$$

Solution We need to compute the integral integrate

$$\mathbb{E}[e^{-rT}(S(T) - K)^+] = e^{-rT}\int_{\frac{1}{\sigma}(\log(K/S(0))-(r-\sigma^2/2)T)}^{\infty}\left(S(0)e^{(r-\sigma^2/2)T+\sigma x} - K\right)\frac{e^{-x^2/(2T)}}{\sqrt{2\pi T}}dx.$$

Substituting $y = x/\sqrt{T}$ and using linearity of the integral the above becomes

$$S(0)e^{-\sigma^2 T/2} \int_{\frac{1}{\sigma\sqrt{T}}(\log(K/S(0))-(r-\sigma^2/2)T)}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2+\sigma\sqrt{T}y} dy$$
$$-Ke^{-rT} \int_{\frac{1}{\sigma\sqrt{T}}(\log(K/S(0))-(r-\sigma^2/2)T)}^{\infty} \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy =$$
$$S(0) \int_{\frac{1}{\sigma\sqrt{T}}(\log(K/S(0))-(r-\sigma^2/2)T)-\sigma\sqrt{T}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz - Ke^{-rT}\Phi(d_-(T, S(0)) =$$
$$S(0)\Phi(d_+(T, S(0)) - Ke^{-rT}\Phi(d_-(T, S(0)).$$