Stochastic Calculus and Black-Scholes Theory MTH772P
Exercises – Sheet 1

1. For $\xi_1, \xi_2, \cdots$ i.i.d. with $\mathbb{P}(\xi_i = \pm 1) = 1/2$ define the discrete-time random walk

$$W_0 = 0, \quad W_n = \xi_1 + \ldots + \xi_n.$$

(i) Formulate and prove the property of independence of increments of $(W_n, n \geq 0)$.

(ii) Show that $(W_n, n \geq 0)$ is a discrete-time Markov chain.

(iii) Show that $(W_n, n \geq 0)$ is a discrete-time martingale.

(iv) Calculate the covariance $\mathrm{Cov}(W_i, W_j)$.

(v) Find the limit distribution of $W_{\lfloor nt \rfloor}/\sqrt{n}$ as $n \to \infty$, for $t > 0$.

2. Let $B = (B(t), t \geq 0)$ be a standard Brownian motion. Show that the following processes are standard BM:

(i) $X(t) = B(t+s) - B(s)$, where $s \geq 0$ is constant.

(ii) $X(t) = B(ct)/\sqrt{c}$, for any $c > 0$,

(iii) $X(t) = B(1-t) - B(1)$, where $t \in [0,1]$ (this BM is defined on $[0,1]$).

3. For X a random variable with density function f consider the event $A = \{X \leq 0\}$.

(i) Define $\mathcal{G}$ to be the σ-algebra generated by A (i.e. the smallest σ-algebra containing event A). Write down the list of all elements of the σ-algebra $\mathcal{G}$.

(ii) In terms of integrals with density f, describe the random variable $\mathbb{E}(X^3|\mathcal{G})$.

(iii) Using the formulas you derived in (ii) show explicitly that $\mathbb{E}(\mathbb{E}(X^3|\mathcal{G})) = \mathbb{E}(X^3)$.

(iv) Make the calculations for (ii), (iii) assuming that X has $\mathcal{N}(0,1)$ distribution.

4. Let $(\mathcal{F}_t, t \geq 0)$ be a filtration for BM. That means that $\{\omega \in \Omega : B(s) \leq x\} \in \mathcal{F}_t$ for $s \leq t$ and $x \in \mathbb{R}$, and that the increments of BM after t are independent of $\mathcal{F}_t$.
For $0 < a < b < c$ show that $B(c) - B(b)$ is independent of $\mathcal{F}_a$.

5. (BM with drift) Let $X(t) = B(t) + t\mu$. Show that $(X(t), t \geq 0)$ is a Markov process and find its transition density. Is the process a martingale?

6. (Geometric BM) Let $S(t) = S(0)\exp(\nu t + B(t))$, where $S(0), \sigma, \nu$ are positive constants. Show that $(S(t), t \geq 0)$ is a Markov process and find its transition density.

7. (Black-Scholes formula) Let $S(t) = S(0)\exp((r - \sigma^2/2)t + B(t))$, where $S(0), \sigma, r$ are positive constants. For $K > 0$ and $T > 0$ show that

$$\mathbb{E}[e^{-rT}(S(T) - K)^+] = S(0)\Phi(d_+(T, S(0))) - Ke^{-rT}\Phi(d_-(T, S(0))),$$

where Φ is the standard normal distribution function, and

$$d_\pm(T, S(0)) = \frac{1}{\sigma\sqrt{T}}\left(\log\frac{S(0)}{K} + (r \pm \frac{\sigma^2}{2})T\right).$$